

MAXIMAL NEWTON POINTS AND THE QUANTUM BRUHAT GRAPH

ELIZABETH MILIĆEVIĆ

ABSTRACT. We discuss a surprising relationship between the partially ordered set of Newton points associated to an affine Schubert cell and the quantum cohomology of the complex flag variety. The main theorem provides a combinatorial formula for the unique maximum element in this poset in terms of paths in the quantum Bruhat graph, whose vertices are indexed by elements in the finite Weyl group. Key to establishing this connection is the fact that paths in the quantum Bruhat graph encode saturated chains in the strong Bruhat order on the affine Weyl group. This correspondence is also fundamental in the work of Lam and Shimozono establishing Peterson’s isomorphism between the quantum cohomology of the finite flag variety and the homology of the affine Grassmannian. One important geometric application of the present work is an inequality which provides a necessary condition for non-emptiness of certain affine Deligne-Lusztig varieties in the affine flag variety.

1. INTRODUCTION

This paper investigates connections between the geometry and combinatorics in two different, but surprisingly related contexts: certain subvarieties of the affine flag variety in characteristic $p > 0$ and the quantum cohomology of the complex flag variety. The main results establish explicit relationships among fundamental questions in both theories, using paths in the quantum Bruhat graph as the primary dictionary. We begin with a brief historical survey of these two geometric contexts in order to frame the informal statement of the main theorem in Section 1.3.

1.1. Newton polygons. In the 1950s, Dieudonné introduced the notion of isocrystals over perfect fields of characteristic $p > 0$ (see [Man63]), which Grothendieck extended to families of F -crystals in [Gro74]. Isogeny classes of F -crystals are indexed by combinatorial objects called *Newton polygons*, a partially ordered set of lattice polygons in the plane. Kottwitz used the machinery of algebraic groups to explicitly study the set of Newton points associated to any connected reductive group G over a discretely valued field in [Kot85] and [Kot97]. In particular, he observed that there is a natural bijection between the set of Frobenius-twisted conjugacy classes in G and a suitably generalized notion of the set of Newton polygons. The poset of Newton points in the context of reductive group theory has interesting combinatorial and Lie-theoretic interpretations, which were first described in [Cha00]. For example, Chai proves that the poset of Newton points is ranked; *i.e.*, any two maximal chains have the same length. In addition to the classification of F -crystals and Frobenius-twisted conjugacy classes, modern interest in the poset of Newton points is motivated by geometric applications to the study of two important families of varieties in arithmetic algebraic geometry: affine Deligne-Lusztig varieties and Shimura varieties; see [GHKR10] and [Rap05].

1.2. Quantum cohomology. Independently, physicists working in the field of superstring theory in the early 1990s observed that certain algebraic varieties seemed to present natural vacuum solutions to superstring equations, and thus developed a theory of *quantum cohomology*; see [Wit95]. Using the notion of mirror symmetry, they were able to employ this cohomological framework to

2010 *Mathematics Subject Classification.* Primary 20G25, 11G25; Secondary 20F55, 14N15, 06A11.

Key words and phrases. Newton polygon, affine Weyl group, affine Deligne-Lusztig variety, Mazur’s inequality, flag variety, quantum cohomology, quantum Bruhat graph.

The author was partially supported by Collaboration Grant 318716 from the Simons Foundation, Grant DP130100674 from the Australian Research Council, and the Max-Planck-Institut für Mathematik.

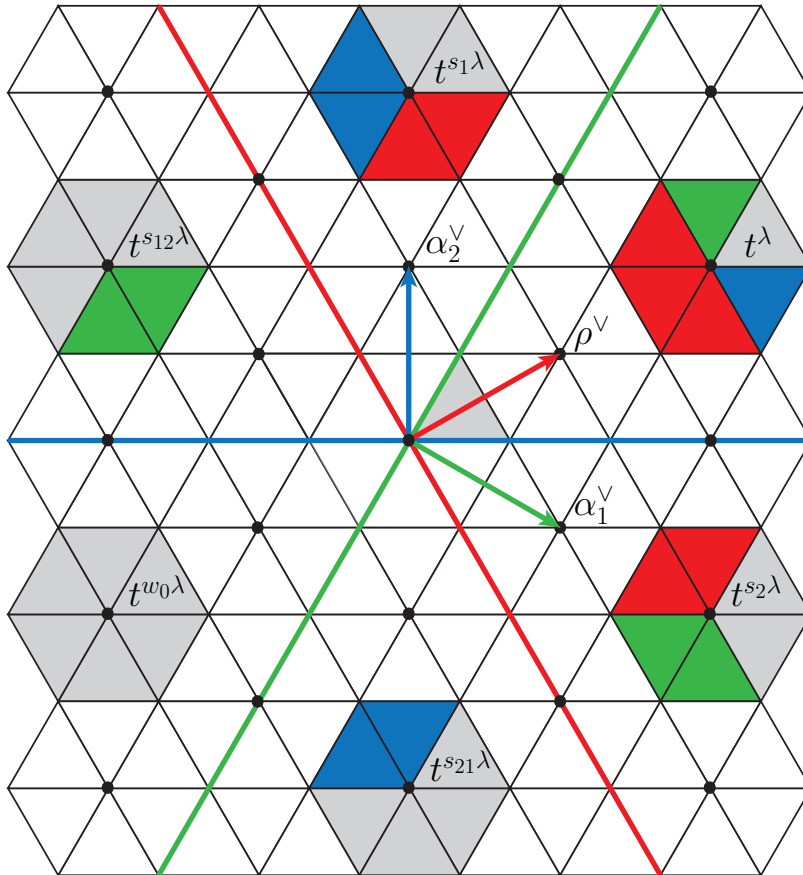


FIGURE 1. Compute the maximum Newton point for $x = t^{v\lambda}w$ by subtracting the coroot of the same color from λ . A gray alcove requires no correction factor.

calculate the number of rational curves of a given degree on a general quintic hypersurface in projective 4-space; see [CdLOGP91]. Mathematicians first rigorously worked out the structure of the quantum cohomology ring for the Grassmanian variety of k -planes in \mathbb{C}^n , and initial mathematical applications were also to enumerative geometry; see [BDW96], [Ber97], and [ST97]. Modern mathematical interest focuses on concretely understanding the structure of the quantum cohomology ring for any homogeneous variety G/P , where G is a complex reductive algebraic group and P a parabolic subgroup. More precisely, the ring $QH^*(G/P)$ has a basis of Schubert classes, indexed by elements of the corresponding Weyl group. The driving question in the field is to find non-recursive, positive combinatorial formulas for expressing the quantum product of two Schubert classes in terms of this basis. Immediate applications include statistics about mapping projective curves to G/P satisfying various incidence conditions, but the impact now extends beyond enumerative geometry into many other aspects of algebraic geometry, combinatorics, representation theory, number theory, and also back to physics.

1.3. Main theorem and applications. The main result in this paper provides a closed combinatorial formula for the maximum element in the poset of Newton points associated to a fixed affine Schubert cell. We express this formula in terms of paths in the *quantum Bruhat graph*, a directed graph with vertices indexed by the finite Weyl group and weights given by the reflections used to get from one element to the other; see Figure 3 and Section 3.1 for an example and the formal definition. We illustrate the main theorem in the case of $G = SL_3$ in Figure 1, and we informally state our main result below. For the precise statement, see Theorem 3.2 and Corollary 3.3.

Theorem. *Let $x = t^{v\lambda}w$ be an element of the affine Weyl group, where w denotes the finite part of x , and v the Weyl chamber in which x lies. Suppose that x is suitably far from the walls of any Weyl chamber. Then the maximum Newton point in the affine Schubert cell indexed by x is given by taking the coroot λ and subtracting the weight of any path of minimal length in the quantum Bruhat graph from $w^{-1}v$ to v .*

This result answers a question about Newton points, which are connected to affine Deligne-Lusztig varieties, but the proof makes heavy use of the tools developed to study quantum Schubert calculus. As such, we are also able to prove corollaries in each of the two fields, albeit also under the hypothesis that x lies suitably far from the walls of any Weyl chamber. Corollary 3.5 illustrates that our question about the maximum Newton point is equivalent to determining the minimum monomial occurring in any quantum product of two Schubert classes in G/B over \mathbb{C} . On the affine side, Corollary 3.6 proposes an analog of Mazur’s inequality for the affine flag variety, placing a sharp upper bound on the Newton point for b in the context in which the affine Deligne-Lusztig variety $X_x(b)$ is non-empty.

This paper represents the full version of extended abstract [Bea12], which was published in the proceedings of the 24th International Conference on “Formal Power Series and Algebraic Combinatorics”. Since that announcement, the statements have been sharpened, and the conventions were also changed to make clearer the correspondence to alcoves in the affine hyperplane arrangement. Spatial constraints on the extended abstract permitted only a brief outline of the proof, and so this version contains all of the details and a full discussion of the subtleties involved in the argument.

1.4. Future directions. The poset of Newton points associated to the affine Schubert cell indexed by $x = t^{v\lambda}w$ perfectly detects the non-emptiness pattern for any affine Deligne-Lusztig variety $X_x(b)$. The main theorem thus says that the quantum Bruhat graph senses non-emptiness of $X_x(b)$ when the element $b \in G(F)$ has the largest possible Newton point. Jointly with Schwer and Thomas in [MST15], the author has developed machinery involving labeled folded alcove walks and root operators which is effective for predicting non-emptiness for elements b with Newton points which lie below $\lambda - 2\rho$ and have integral slopes. Although each of the two techniques presents some challenges when x lies outside of the “shrunk” Weyl chambers, one might hope that an interpolation between these two methods will result in a full picture for the non-emptiness problem.

More mystifying are the connections which arise between the main theorem and quantum Schubert calculus. Although Peterson established an isomorphism between suitable localizations of the equivariant homology of the affine Grassmannian and the equivariant quantum cohomology of the complete flag variety [Pet96], it remains to explain the precise relationship of these cohomology theories to the geometry of other subvarieties of the affine flag variety, especially in characteristic $p > 0$. For example, this paper shows that the maximal element of the poset of Newton points $\mathcal{N}(G)_x$ and the minimal monomial q^d in the quantum product of two Schubert classes are dictated by exactly the same combinatorial information—it would therefore be natural to explore whether the posets share other elements besides these extrema.

Since Lam and Shimozono’s proof of the Peterson isomorphism in [LS10], many other applications of the “quantum equals affine” phenomenon have been discovered. We highlight several results beyond Schubert calculus which have made similar critical use of the connection between alcove walks and the quantum Bruhat graph. In a series of papers, Lenart, Naito, Sagaki, Schilling, and Shimozono compute the energy function on tensor products of certain Kirillov-Reshetikhin crystals in terms of the parabolic quantum Bruhat graph; see [LNS⁺15] and the references therein. Feigin and Makedonskyi describe the representation theory of generalized Weyl modules in terms of a generating function on quantum alcove paths [FM15], based on a similar formula of Orr and Shimozono for a specialization of nonsymmetric Macdonald polynomials [OS13]. Most recently, Naito and Watanabe proved a combinatorial formula for periodic R -polynomials in terms of paths in a doubled quantum Bruhat graph [WN16]. These R -polynomials can be used to compute periodic

Kazhdan-Lusztig polynomials, which conjecturally determine the characters of irreducible modules of a reductive group over a field of positive characteristic. It would be interesting to understand the geometric and/or representation-theoretic phenomena which cause the answers to these seemingly different questions to be governed by the same combinatorics.

Finally, it is our hope that the combinatorics community might invigorate new interest in the poset of Newton points studied here, perhaps at least in the case of $G = GL_n$ in which the setup is quite combinatorial. For this reason, we make an effort to be very concrete throughout Section 2 in our discussion of the Newton map, mentioning open problems along the way. Regarding the poset $\mathcal{N}(G)_x$ of Newton points associated to an affine Weyl group element x , besides the results in this paper on maximal elements, some information about minimal elements is known [GHN15], as well as some of its integral elements [MST15]. Beyond groups of low rank, however, we have not yet established even the most basic desirable poset properties, such as whether or not it is ranked, a lattice, shellable, or for which x it is a subinterval of the poset of all Newton points for G . Of course, the ideal goal would then be to apply such poset combinatorics back to the geometry of the associated varieties in characteristic $p > 0$.

1.5. Overview of the paper. This paper is written with two potentially disjoint audiences in mind: those interested in Newton points and those interested in quantum Schubert calculus. After establishing notation, we thus open in Section 2 with an elementary review of Newton polygons, presenting explicit formulas in many special cases along with examples. The reader familiar with reductive groups over local fields and the Newton map can safely skip to Section 2.7 for a refresher on the maximum Newton point. Section 3 begins with a review of the quantum Bruhat graph and its main combinatorial properties. Experts in quantum Schubert calculus can move straight into Section 3.2, which contains a precise statement of the main result as Theorem 3.2, continuing directly to Section 3.3 for the main application to quantum cohomology. The main application to affine Deligne-Lusztig varieties is then presented and proved in Section 3.4.

The remainder of the paper is dedicated to the proof of Theorem 3.2. We start by invoking an alternative combinatorial formula of Viehmann for the maximum Newton point from [Vie14], which is expressed in terms of both affine Bruhat order and dominance order; see Theorem 2.10. The connection to quantum Schubert calculus is then made by using an observation of Lam and Shimozono in their proof of the Peterson isomorphism [LS10], which places covering relations in affine Bruhat order in two-to-one correspondence with edges in the quantum Bruhat graph. In the remainder of Section 4, we iterate Proposition 4.2, stitching the relations from this correspondence together to form saturated chains. Section 5 then lays the groundwork to compare the Newton points for all of the elements lying below a given x in Bruhat order. This section represents the technical heart of the paper, involving careful combinatorics on root hyperplanes in order to bound the maximum Newton point from below. We put the resulting sequence of lemmas to work in Section 6, making the primary reduction step to considering only pure translations less than x . The proof of the main theorem then follows immediately in Section 6.2. We conclude in Section 6.3 with a discussion of the role of the superregularity hypothesis in the proof of Theorem 3.2.

Acknowledgments. The author thanks John Stembridge and Thomas Lam for instrumental conversations during the conception and development of this project. Part of this work was conducted during a visit to the University of Melbourne, and the author thanks Arun Ram for fostering an exceptionally mathematically stimulating environment. This paper was completed during the a stay at the Max-Planck-Institut für Mathematik, and the author wishes to gratefully acknowledge the institute for its excellent working conditions.

2. THE POSET OF NEWTON POLYGONS

After we establish some basic notation for root systems and Weyl groups, the primary purpose of this section is to introduce the Newton map and discuss some of the basic properties of partially

ordered sets of Newton points. The formal definition is fairly abstract, and so we offer the reader three alternative ways to think about constructing the Newton polygon for an element of $GL_n(F)$. We discuss the most important special case alongside the general definition in Section 2.4. Each method we review poses its own computational challenges, but an implicit common theme is the need to choose a suitable representative of each σ -conjugacy class in $G(F)$. We do not intend to provide a comprehensive survey of this subject, nor do we provide an exhaustive list of constructions for the Newton point, even in the special case of GL_n . Rather, our goal is to provide some general exposure to the methods, particularly for the audience more interested in the combinatorial aspects.

2.1. Notation. Let G be a split connected reductive group, and B a fixed Borel subgroup with T a maximal torus in B . Let $R = R^+ \sqcup R^-$ be the set of roots, which can be viewed as a subset of the group of characters $X^*(T)$. The set $\Delta = \{\alpha_i \mid i \in I\}$ is an ordered basis of simple roots in $X^*(T)$, and $\{\alpha_i^\vee \mid i \in I\}$ a basis for R^\vee of simple coroots in $X_*(T)$, which are dual with respect to the pairing $\langle \cdot, \cdot \rangle : X_*(T) \times X^*(T) \rightarrow \mathbb{Z}$. The parabolic subgroups P containing B are in bijection with subsets Δ_P of Δ , each of which has an associated root system R_P consisting of the roots of the Levi factor. The finite Weyl group W is the quotient $N_G(T)/T$. Denote by r_α the reflection in W corresponding to the positive root $\alpha \in R^+$, and then write s_i for the simple reflection in W corresponding to the simple root $\alpha_i \in \Delta$. There is a natural action of W on R which induces a permutation on the set of reflections in W via $r_{v\alpha} = vr_\alpha v^{-1}$ for any $\alpha \in R^+$ and $v \in W$. Define ρ to be the half-sum of the positive roots.

Denote by $Q = \bigoplus \mathbb{Z}\alpha_i$ and $Q^\vee = \bigoplus \mathbb{Z}\alpha_i^\vee$ the root and coroot lattices, respectively. We also occasionally need the coweight lattice $P^\vee = \bigoplus \mathbb{Z}\omega_i^\vee \subset X_*(T)$ spanned by the fundamental coweights ω_i^\vee , which are dual to the simple roots α_i . The finite Weyl group W acts on $\mathbb{R}^r \cong Q^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ as a finite reflection group, where r is the rank of G . Denote by H_α the hyperplane in \mathbb{R}^r orthogonal to the root α , which is the reflecting hyperplane corresponding to r_α in this representation. We say that $\lambda \in P^\vee$ is dominant if $\langle \lambda, \alpha_i \rangle \geq 0$ for all $\alpha_i \in \Delta$. Denote by Q^+ and P^+ the set of dominant elements of Q^\vee and P^\vee , respectively. By λ^+ we mean the unique dominant coroot in the W -orbit of $\lambda \in Q^\vee$. More generally, define the (closed) dominant Weyl chamber to be

$$(2.1) \quad C = \{\lambda \in \mathbb{R}^r \mid \langle \lambda, \alpha \rangle \geq 0, \forall \alpha \in R^+\}.$$

Dominance order forms a natural partial ordering on the coroot lattice. Given $\lambda, \mu \in Q^\vee$, we say that $\lambda \geq \mu$ if and only if $\lambda - \mu$ is a nonnegative linear combination of positive coroots. We say that $\lambda \in Q^\vee$ is regular if the stabilizer of λ in W is trivial. Following [LS10], the coroot λ is said to be *superregular* if $\langle \lambda, \alpha \rangle \geq M$ for all $\alpha \in R^+$, for some sufficiently large M .

In the context of the Newton map, we will work with a split connected reductive group G over the discretely valued field $F = \overline{\mathbb{F}_q}((t))$ for $q = p^s$, which has characteristic $p > 0$. We denote the discrete valuation by $\text{val} : F \rightarrow \mathbb{Z}$, and this map picks out the smallest power of t occurring with nonzero coefficient in the Laurent series; for our purposes we define $\text{val}(0) = -\infty$. The ring of integers in F is given by the ring of formal power series $\mathcal{O} = \overline{\mathbb{F}_q}[[t]]$. For any $\mu \in X_*(T)$, let t^μ denote the image of t under $\mu : \mathbb{G}_m \rightarrow T$. We can extend the usual Frobenius automorphism $x \mapsto x^q$ on $\overline{\mathbb{F}_q}$ to a map $\sigma : F \rightarrow F$ by defining σ to act on the coefficients: $\sum a_i t^i \mapsto \sum a_i^q t^i$. Two elements $g_1, g_2 \in G(F)$ are said to be σ -conjugate if there exists an $h \in G(F)$ such that $hg_1\sigma(h)^{-1} = g_2$.

The affine Weyl group of $G(F)$ is isomorphic to the semi-direct product $\widetilde{W} = Q^\vee \rtimes W$, and any $x \in \widetilde{W}$ may be written as $x = t^\lambda w$ for some $\lambda \in Q^\vee$ and $w \in W$. If λ is regular, then there exists a unique $v \in W$ such that $t^\lambda w = t^{v\lambda^+} w$. The affine Weyl group is generated by the following affine reflections on \mathbb{R}^r :

$$(2.2) \quad r_{\alpha,m}(\lambda) = \lambda - (\langle \lambda, \alpha \rangle - m)\alpha^\vee.$$

The element $r_{\alpha,m}$ is then identified with the reflection across the affine hyperplane

$$(2.3) \quad H_{\alpha,m} = \{\lambda \in \mathbb{R}^r \mid \langle \lambda, \alpha \rangle = m\}.$$

Note that $r_{\alpha,0} = r_\alpha$ and $H_{\alpha,0} = H_\alpha$. The connected components of $\mathbb{R}^r \setminus \{H_{\alpha,m} \mid \alpha \in R^+, m \in \mathbb{Z}\}$ are called alcoves, and we use freely the natural bijection between elements of the affine Weyl group and alcoves. In terms of the isomorphism $\widetilde{W} = Q^\vee \rtimes W$, we can write $r_{\alpha,m} = t^{m\alpha^\vee} r_\alpha$, which acts by left multiplication as the reflection across $H_{\alpha,m}$. Because each affine reflection is also associated to a unique positive root in the affine Lie algebra, we typically write r_β rather than $r_{\alpha,m}$ for brevity. It should always be clear from context whether r_β represents a linear or an affine reflection, especially because we typically reserve the letters v, w for finite Weyl group elements and x, y for affine Weyl group elements. Finally, denote by $\ell : \widetilde{W} \rightarrow \mathbb{Z}_{\geq 0}$ the length function, and by w_0 the element of longest length in the finite Weyl group.

2.2. Newton polygons via isocrystals. The notion of an isocrystal over a perfect field of characteristic $p > 0$ was introduced by Dieudonné and generalized by Grothendieck [Gro74]. In his classification, Dieudonné proved that isomorphism classes of isocrystals are indexed by Newton polygons [Man63], which then became the starting point for the development of the Newton map in the context of algebraic groups by Kottwitz [Kot85].

Definition 2.1. An *isocrystal* (V, Φ) is a finite-dimensional vector space V over F together with a σ -linear bijection $\Phi : V \rightarrow V$; that is, $\Phi(av) = \sigma(a)\Phi(v)$ for $a \in F$ and $v \in V$.

A simple example of an isocrystal is (F^n, Φ) , where $\Phi = A \circ \sigma$ for some $A \in GL_n(F)$, and σ acts on V coordinate-wise. Conversely, if we fix a basis $\{e_1, \dots, e_n\}$ for V , then note that we can associate a matrix $A \in GL_n(F)$ to (V, Φ) defined by $\Phi(e_i) = \sum_{j=1}^n A_{ji} e_j$, in which case we write $\Phi = A \circ \sigma$ for $A = (A_{ij})$. More generally, for G any connected reductive group over F , the choice of an element $g \in G(F)$ together with a finite-dimensional representation of G determines an isocrystal over F ; see [RR96] for a discussion of F -isocrystals with G -structure following [Kot85].

Dieudonné showed that the category of isocrystals over F is semi-simple, and that the simple objects are naturally indexed by \mathbb{Q} . That is, any isocrystal (V, Φ) is isomorphic to a direct sum $V = \bigoplus_{i=1}^n V_{s_i/r_i}$ where $\gcd(r_i, s_i) = 1$ and the V_{s_i/r_i} are simple objects in the category; for a proof of semisimplicity and a characterization of the simple isocrystals, see [Dem72].

Definition 2.2. We now present our first definition for the Newton polygon.

- (1) If (V, Φ) is an n -dimensional isocrystal over F such that $V = \bigoplus_{i=1}^n V_{s_i/r_i}$, then the *Newton slope sequence* for (V, Φ) is defined to be $\lambda = (\lambda_1, \dots, \lambda_n) \in Q^+ \otimes_{\mathbb{Z}} \mathbb{Q}$ where $\lambda_i = s_i/r_i$, each λ_i is repeated r_i times, and $\lambda_1 \geq \dots \geq \lambda_n$.

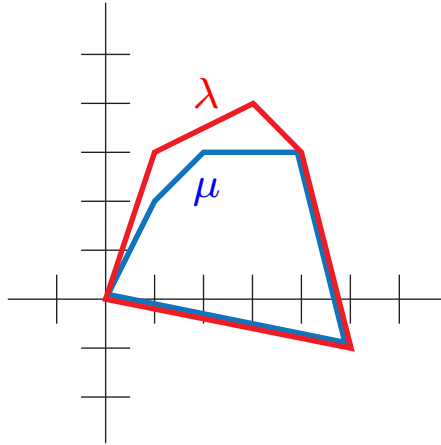


FIGURE 2. A pair of Newton polygons for $GL_5(F)$ with slope sequences $\lambda \geq \mu$.

- (2) The *Newton polygon* for an n -dimensional isocrystal (V, Φ) with slope sequence λ is the graph of the function $\bar{\nu} : [0, n] \rightarrow \mathbb{R}$ given by

$$\begin{cases} \bar{\nu}(i) = 0, & \text{if } i = 0, \\ \bar{\nu}(i) = \lambda_1 + \cdots + \lambda_i, & \text{if } i = 1, \dots, n, \end{cases}$$

and then extended linearly between successive integers.

- (3) If we denote by $\mathcal{N}(G)$ the set of all possible Newton polygons for isocrystals arising from elements in $G(F)$, then $\nu : G \rightarrow \mathcal{N}(G)$ which sends $\nu(g) \mapsto \lambda$ is the *Newton map*.

For example, the red Newton polygon in Figure 2 corresponds to a 5-dimensional isocrystal having Newton slope sequence $\lambda = (3, \frac{1}{2}, \frac{1}{2}, -1, -4)$. Because a Newton slope sequence uniquely determines a Newton polygon and vice versa in the case of $G = GL_n$, we occasionally refer to elements $\nu(g)$ interchangeably as both polygons in the plane and slope sequences in $Q^+ \otimes_{\mathbb{Z}} \mathbb{Q}$.

2.3. Newton polygons via characteristic polynomials. We now provide a more concrete definition of the Newton polygon associated to an element $g \in GL_n(F)$ which only requires basic linear algebra. Although we do not appeal to Dieudonné's classification, the definition presented in this section is in fact equivalent to Definition 2.2.

Define a ring $R := F[\sigma]$ by formally adjoining the Frobenius automorphism. Multiplication in R is non-commutative, defined such that $\sigma a = \sigma(a)\sigma$ for $a \in F$. There exist both a right and left division algorithm, and so R is a principal ideal domain. Given an isocrystal (V, Φ) over F , identifying $\sigma^i v := \Phi^i(v)$ makes V into an R -module. In fact, the Frobenius twist makes (V, Φ) into a cyclic module over the ring R ; that is, $Rv = V$ for some v in V . In this context, we call the generator v a *cyclic vector*. Upon choosing a cyclic vector v , we may thus write $V \cong R/Rf$ for some $f = \Phi(v)^n + a_1\Phi(v)^{n-1} \cdots + a_{n-1}\Phi(v) + a_nv \in R$ with $a_i \in F$, where $n = \dim_F(V)$. We call f the *characteristic polynomial* associated to this isocrystal and cyclic vector (V, Φ, v) .

Definition 2.3. Given an isocrystal and a choice of cyclic vector (V, Φ, v) , define the associated Newton polygon as the result of the following algorithm:

- (1) Find the characteristic polynomial $f = \Phi(v)^n + a_1\Phi(v)^{n-1} \cdots + a_{n-1}\Phi(v) + a_nv \in R$, satisfying $V \cong R/Rf$.
- (2) Plot the set of points $\{(0, 0), (i, \text{val}(a_i)) \mid i = 1, \dots, n\}$ recording the valuations of the coefficients of this characteristic polynomial.
- (3) Take the upper convex hull of this set of points; *i.e.* form the tightest-fitting polygon which passes either through or above all of the plotted points.

While the characteristic polynomial f clearly depends on the choice of a cyclic vector, the Newton polygon associated to (V, Φ, v) is actually independent of v . We can thus safely refer to the result of this construction as the *Newton polygon* for the isocrystal (V, Φ) . By recording the slopes of each edge of the Newton polygon left to right, repeated with multiplicity, we obtain the corresponding Newton slope sequence $\lambda \in Q^+ \otimes_{\mathbb{Z}} \mathbb{Q}$.

We illustrate via example how one can find a suitable choice of a cyclic vector, calculate the characteristic polynomial, and then construct the associated Newton polygon.

Example 2.4. Let (F^2, Φ) be an isocrystal, where $\Phi = g \circ \sigma$ for some $g := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F)$.

The first standard basis vector e_1 is a cyclic vector for (F^2, Φ) if and only if $c \neq 0$, since

$$(2.4) \quad e_1 \wedge \Phi(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} a \\ c \end{pmatrix} = c(e_1 \wedge e_2).$$

If $c \neq 0$, then we can compute the characteristic polynomial for (F^2, Φ, e_1) by solving for α and β in the following F -linear system of equations:

$$(2.5) \quad \Phi^2(e_1) + \alpha \cdot \Phi(e_1) + \beta \cdot e_1 = 0,$$

$$(2.6) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sigma(a) \\ \sigma(c) \end{pmatrix} + \alpha \begin{pmatrix} a \\ c \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Therefore,

$$(2.7) \quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = - \begin{pmatrix} a & 1 \\ c & 0 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sigma(a) \\ \sigma(c) \end{pmatrix} = \begin{pmatrix} -\sigma(a) - \frac{\sigma(c)}{c}d \\ \frac{\sigma(c)}{c}(ad - bc) \end{pmatrix}.$$

Note that if $a, c \in F^\sigma$ so that $\sigma(a) = a$ and $\sigma(c) = c$, then this σ -twisted version of the characteristic polynomial coincides with the usual characteristic polynomial for GL_2 .

For a concrete example, now suppose that $g = \begin{pmatrix} t^2 & t \\ 1 & t^3 \end{pmatrix}$. Using the formula derived above, the coefficients of the characteristic polynomial then equal $\alpha = -t^2 - t^3$ and $\beta = -t + t^5$. Taking valuations, we see that $\text{val}(\alpha) = 2$ and $\text{val}(\beta) = 1$. Therefore, the Newton polygon $\nu(g)$ is the convex hull of the three points $(0, 0)$, $(1, 2)$, and $(2, 1)$ and has slope sequence $\lambda = (2, -1)$.

Using either definition of the Newton polygon presented so far, there are some challenges to explicitly calculating $\nu(g)$, even given a specific element $g \in G(F)$. Definition 2.2 requires a detailed understanding of the simple objects in the category of isocrystals. Although Definition 2.3 is relatively concrete, calculations such as those performed in Example 2.4 can become unwieldy for groups of large rank. To get a sense for how complexity grows with the rank of G , see [Bea09] for a treatment of the case $G = SL_3$, which is the only other group for which this calculation has been fully carried out in the literature.

2.4. Newton points via extended affine Weyl group elements. The most general definition of the Newton map was given by Kottwitz in [Kot85] and [Kot97], in which he characterized its image on the set $B(G)$ of σ -conjugacy classes in $G(F)$. The image of an element $g \in G(F)$ under the Newton map is a σ -conjugacy class invariant, as is the connected component of $G(F)$. Putting the Newton map together with the *Kottwitz homomorphism* κ identifying the connected component, one obtains an injective map

$$(2.8) \quad (\nu, \kappa) : B(G) \hookrightarrow (P^+ \otimes_{\mathbb{Z}} \mathbb{Q}) \times \pi_1(G).$$

Moreover, the restriction of the map $G(F) \rightarrow B(G)$ to the normalizer $(N_G T)(F)$ factors through the *extended affine Weyl group* $\widetilde{W}_e := (N_G T)(F)/T(\mathcal{O}) \cong P^\vee \rtimes W$, and the map $\widetilde{W}_e \rightarrow B(G)$ is surjective; see Corollary 7.2.2 in [GHKR10]. Therefore, in order to define the Newton map on an algebraic group $G(F)$, it suffices to be able to compute its image on elements of the extended affine Weyl group. Both for this reason and for the purpose of our own argument, the following formula for the image of the Newton map on an element in \widetilde{W}_e is the most important special case.

Proposition 2.5. *Let $y = t^\lambda w \in \widetilde{W}_e$, and suppose that the order of w in W equals m . Then the Newton point for y equals*

$$(2.9) \quad \nu(y) = \left(\frac{1}{m} \sum_{i=1}^m w^i(\lambda) \right)^+.$$

Formula (2.9) is a standard fact in the literature; for example, see Section 4.2 in [Gör10].

For general $G(F)$, the image of the Newton map is thus simply an element of $P^+ \otimes_{\mathbb{Z}} \mathbb{Q}$, and from now on we typically refer to $\nu(y)$ as the Newton *point* for y , rather than the Newton *polygon*. As we have seen, when $G = GL_n$ then the Newton point does in fact correspond to the slope sequence for a Newton polygon, and so we use either term when there is no risk of confusion.

We now provide an explicit example illustrating Proposition 2.5, since Equation (2.9) will play such a fundamental role in the proof of Theorem 3.2.

Example 2.6. Let $G = SL_3(F)$ so that $W = S_3$ is generated by two simple reflections s_1 and s_2 . Let $y = t^{(-2,0,2)}s_1 \in \widetilde{S}_3$. Then the Newton point for y equals

$$\nu(y) = \left[\frac{1}{2}((0, -2, 2) + (-2, 0, 2)) \right]^+ = \left[\frac{1}{2}(-2, -2, 4) \right]^+ = (2, -1, -1).$$

The corresponding Newton polygon is the convex hull of the three points $(0, 0)$, $(1, 2)$, and $(3, 0)$.

In proving our main theorem we will only ever need to compute $\nu(y)$ for elements $y \in \widetilde{W}$. Therefore, for all practical purposes, Equation (2.9) will serve as our definition of the Newton map for general $g \in G(F)$; in practice, the only additional required step is to start by finding an element of $(N_G T)(F)$ which is σ -conjugate to g .

2.5. The partial ordering on the set of Newton points. There is a natural partial ordering on the set $\mathcal{N}(G)$ of Newton points occurring for elements in $G(F)$. We compare two Newton points $\lambda, \mu \in P^+ \otimes_{\mathbb{Z}} \mathbb{Q}$ by extending the dominance order on P^\vee to \mathbb{Q}^r . Namely, we say that $\lambda \geq \mu$ if and only if $\lambda - \mu$ is a nonnegative rational combination of positive coweights. In Section 5.1, we discuss another useful interpretation of the partial ordering on $\mathcal{N}(G)$ in terms of a convexity condition on Weyl orbits of rational points in the closed Weyl chambers.

Example 2.7. If $\lambda = (3, \frac{1}{2}, \frac{1}{2}, -1, -4)$ is the slope sequence for the red Newton polygon from Figure 2, then λ is greater than the blue Newton point $\mu = (2, 1, 0, 0, -4)$, since $\lambda - \mu = \alpha_1^\vee + \frac{1}{2}\alpha_2^\vee + \alpha_3^\vee$. Equivalently, all partial sums of the form $\lambda_1 + \dots + \lambda_i$ are greater than or equal to those for μ .

Given a Newton polygon λ in the plane, we say that another Newton polygon μ satisfies $\mu \leq \lambda$ if they share a left and rightmost vertex and all edges of λ lie either on or above those of μ . Compare the red and blue Newton polygons in Figure 2, which illustrates that dominance order coincides with containment of Newton polygons in the case of $G = GL_n$.

The full poset of Newton polygons $\mathcal{N}(G) = \{\nu(g) \mid g \in G\}$ was initially studied in [RR96], [Kot97], and [Cha00] from the perspective of arithmetic algebraic geometry. In particular, Chai established that $\mathcal{N}(G)$ has many desirable combinatorial properties of partially ordered sets. For example, he proves that $\mathcal{N}(G)$ is a *ranked* poset; *i.e.* all maximal chains have the same length, and he also shows that $\mathcal{N}(G)$ is a lattice.

2.6. Newton points in affine Schubert cells. In the context of a reductive group over a local field, one version of the Bruhat decomposition says that

$$(2.10) \quad G(F) = \bigsqcup_{x \in \widetilde{W}} IxI,$$

where the *Iwahori subgroup* I is defined to be the inverse image of B under the evaluation map $G(\overline{\mathbb{F}_q}[[t]]) \rightarrow G(\overline{\mathbb{F}_q})$ sending $t \mapsto 0$. For example, if $G = SL_n(F)$, then

$$(2.11) \quad I = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} & \dots & \mathcal{O} \\ t\mathcal{O} & \mathcal{O}^\times & \dots & \mathcal{O} \\ \vdots & \vdots & \ddots & \vdots \\ t\mathcal{O} & t\mathcal{O} & \dots & \mathcal{O}^\times \end{pmatrix}.$$

Motivated by applications to Shimura varieties and affine Deligne-Lusztig varieties, it is useful to study the combinatorics of subsets of $\mathcal{N}(G)$ in which one restricts to Newton points which arise from a fixed cell in this affine Bruhat decomposition. The question then becomes to study the set

$$(2.12) \quad \mathcal{N}(G)_x := \{\nu(g) \mid g \in IxI\}$$

of Newton points occurring for elements in the *affine Schubert cell* IxI . These subsets clearly inherit the partial ordering on $\mathcal{N}(G)$.

The posets $\mathcal{N}(G)_x$ have only been fully characterized for groups of low rank and/or when x has a special form, but many nice combinatorial properties of $\mathcal{N}(G)$ also hold for $\mathcal{N}(G)_x$ in these cases. For example, in [Bea09] the author proves that if $G = GL_2$ or $G = SL_3$, then the poset $\mathcal{N}(G)_x$ is a ranked lattice. For another special case, if $x = t^\lambda$, then $\mathcal{N}(G)_x = \{\lambda^+\}$ is a single element set; see Corollary 9.2.1 in [GHKR10]. The precise relationship between $\mathcal{N}(G)_x$ and $\mathcal{N}(G)$ remains quite opaque in general. For example, outside of these same special cases, it is not known for which x the poset $\mathcal{N}(G)_x$ is a subinterval of $\mathcal{N}(G)$.

We remark that if we use a different decomposition on $G(F)$ than the Bruhat decomposition from (2.10), then the study of the corresponding posets of Newton points can become much simpler. For example, using the Cartan decomposition into double cosets of the maximal compact subgroup $K = G(\mathcal{O})$, the Newton poset for elements in $Kt^\lambda K$ consists of all $\mu \in Q^+ \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\mu \leq \lambda^+$, subject to the integrality condition that the denominator of any rational slope divides its multiplicity; *i.e.* in this case one always obtains a full subinterval of $\mathcal{N}(G)$.

2.7. Maximal Newton points. While the poset $\mathcal{N}(G)_x$ remains rather mysterious in many ways, we review the well-known fact that it does possess a unique maximum element.

Claim 2.8. *The poset $\mathcal{N}(G)_x$ contains a unique maximum element.*

Proof. For a fixed $x \in \widetilde{W}$, the double coset IxI is irreducible. Denote by $G_\lambda = \{g \in G \mid \nu(g) = \lambda\}$, and consider the intersections $(IxI)_\lambda := G_\lambda \cap IxI$. Then IxI is the finite union of subsets of the form $(IxI)_\lambda$, any two of which are disjoint. If $\lambda \in \mathcal{N}(G)_x$ is maximal, then $(IxI)_\lambda$ is an open subset of IxI . But since IxI is irreducible, there must exist a unique maximum element in $\mathcal{N}(G)_x$. \square

Definition 2.9. Given $x \in \widetilde{W}$, we define the *maximum Newton point* $\nu_x \in Q^+ \otimes_{\mathbb{Z}} \mathbb{Q}$, to be the unique maximum element in $\mathcal{N}(G)_x$; *i.e.* we have $\nu_x \geq \lambda$ for all $\lambda \in \mathcal{N}(G)_x$.

The first closed formula for the unique maximum element ν_x in $\mathcal{N}(G)_x$ was discovered by Viehmann, who reduced the computation to an interplay between the combinatorics of two natural partial orderings associated to elements of the affine Weyl group.

Theorem 2.10 (Corollary 5.6 [Vie14]). *The maximum Newton point associated to $x \in \widetilde{W}$ is*

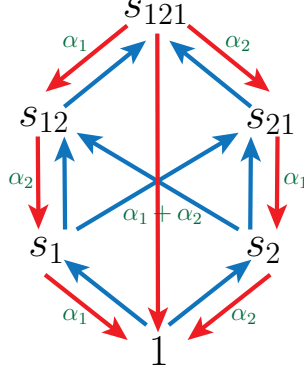
$$(2.13) \quad \nu_x = \max\{\nu(y) \mid y \in \widetilde{W}, y \leq x\},$$

where the maximum is taken with respect to dominance order and the elements y and x are related by Bruhat order.

Although elegant, in practice this formula is difficult to implement without the aid of a computer except in certain special cases, involving first finding every element less than the fixed affine Weyl group element x in Bruhat order, and further computing and comparing the Newton points for each of those elements. Our main theorem, which is formally stated in Section 3.2, provides a combinatorial formula for ν_x which may be easily computed by hand.

3. THE QUANTUM BRUHAT GRAPH AND APPLICATIONS

The main result in this paper shows that there is a closed combinatorial formula for the maximum element in the poset of Newton points in terms of paths in the *quantum Bruhat graph*. The nomenclature comes from the fact that this graph was introduced by Brenti, Fomin, and Postnikov in [BFP99] to capture the multiplicative structure of the quantum cohomology ring of the complex flag variety, in particular the Chevalley-Monk rule for multiplying by a divisor class.


 FIGURE 3. The quantum Bruhat graph for S_3 .

3.1. The quantum Bruhat graph. We now formally define the quantum Bruhat graph, which will be our primary combinatorial tool. The vertices are given by the elements of the finite Weyl group $w \in W$. Two elements are connected by an edge if they are related by a reflection satisfying one of two “quantum relations.” More precisely, there is a directed edge $w \rightarrow wr_\alpha$ if one of two length relationships between w and wr_α is satisfied:

$$\begin{aligned} w &\xrightarrow{\text{blue}} wr_\alpha \text{ if } \ell(wr_\alpha) = \ell(w) + 1, \text{ or} \\ w &\xrightarrow{\text{red}} wr_\alpha \text{ if } \ell(wr_\alpha) = \ell(w) - \langle \alpha^\vee, 2\rho \rangle + 1. \end{aligned}$$

The first type of edges are simply those corresponding to covers in the usual Hasse diagram for the strong Bruhat order on W . The second type of edges, all of which are directed downward in the graph, are “quantum” edges coming from the quantum Chevalley-Monk formula of [Pet96]. The edges are then labeled by the root corresponding to the reflection used to get from one element to the other, so that the edge from $w \rightarrow wr_\alpha$ is labeled by α . Figure 3 shows the quantum Bruhat graph for $W = S_3$, in which we abbreviate $s_i s_j$ simply as s_{ij} .

We now define the weight of any path in the quantum Bruhat graph. For an edge $w \xrightarrow{\text{blue}} wr_\alpha$ resulting from the relation $\ell(wr_\alpha) = \ell(w) + 1$, there is no contribution to the weight. On the other hand, an edge $w \xrightarrow{\text{red}} wr_\alpha$ arising from the relation $\ell(wr_\alpha) = \ell(w) - \langle \alpha^\vee, 2\rho \rangle + 1$ contributes a weight of α^\vee . The *weight of a path in the quantum Bruhat graph* is then defined to be the sum of the weights of the edges in the path. For example, in Figure 3, the weight of any of the three paths of minimal length from s_{12} to s_2 , all of which have length 3, equals $\alpha_1^\vee + \alpha_2^\vee$.

It will also sometimes be convenient to record the weight of a path in the quantum Bruhat graph as a vector in \mathbb{Z}^r , where r is the rank of G . If we express the weight $\mu \in Q^\vee$ of a path in terms of the basis of simple coroots, say $\mu = d_1 \alpha_1^\vee + \cdots + d_r \alpha_r^\vee$, then we define $d = (d_1, \dots, d_r) \in \mathbb{Z}^r$ and equivalently also refer to the vector d as the weight of the path. The purpose of this alternative is that we may then associate monomials in a certain set of commuting variables q_1, \dots, q_r to each path. In particular, denote by q^d the monomial $q_1^{d_1} \cdots q_r^{d_r}$; see Section 3.3 for the motivation.

We now record several combinatorial statements about paths in the quantum Bruhat graph from [Pos05], whose proofs rely on the combinatorics of the tilted Bruhat order introduced in [BFP99].

Proposition 3.1 (Lemma 1, Theorem 2 [Pos05]). *Let $u, v \in W$ be any Weyl group elements.*

- (1) *There exists a directed path from u to v in the quantum Bruhat graph.*
- (2) *The length of a shortest path between any two elements in the quantum Bruhat graph equals $\ell(w)$ for some element $w \in W$.*
- (3) *All shortest paths from u to v have the same weight, say d_{\min} .*
- (4) *If d is the weight of any path from u to v , then q^d is divisible by $q^{d_{\min}}$.*

We use a special case of property (3) concerning the uniqueness of the weight of a minimal length path in the proof of Theorem 3.2, although we point out that an independent proof of the final step in Proposition 6.1 would provide a *geometric* explanation for each of these combinatorial properties. Further, using the automorphisms of the Iwahori subgroup discussed in [Bea09], the author expects that should also be possible to provide independent geometric proofs of the symmetries of the quantum Bruhat graph for $W = S_n$ appearing in [Pos01], which in turn correspond to symmetries of Gromov-Witten invariants.

3.2. Statement of the main theorem. We are now prepared to formally state our main result, which provides a readily computable combinatorial formula for the maximum Newton point ν_x in $N(G)_x$. The general shape of ν_x for $x = t^{v\lambda}w$ with λ dominant is that $\nu_x = \lambda - \mu$, where μ is a correction factor obtained by looking at the weight of any minimal length path in the quantum Bruhat graph between two vertices uniquely determined by the pair of finite Weyl group elements associated to x .

The main theorem requires a superregularity hypothesis on the coroot λ , which we make precise in Definition 5.10. We introduce a constant $M_k \in \mathbb{Z}_{\geq 0}$ which depends on a nonnegative integer k uniquely determined by the pair $w, v \in W$, as well as the Lie type of G . The superregularity hypothesis stated in Theorem 3.2 is the sharpest that the current method of proof permits; see Section 6.3 for a detailed discussion of the components of the proof which introduce these superregularity conditions. Corollary 3.3 below presents the same result with a stronger, but more uniform superregularity hypothesis.

Theorem 3.2. *Let $x = t^{v\lambda}w \in \widetilde{W}$, and consider any path of minimal length k from $w^{-1}v$ to v in the quantum Bruhat graph for W . If $\langle \lambda, \alpha_i \rangle > M_k$ for all simple roots $\alpha_i \in \Delta$, then the maximum Newton point associated to x equals*

$$(3.1) \quad \nu_x = \lambda - \alpha_x^\vee,$$

where α_x^\vee is the weight of the chosen path from $w^{-1}v$ to v .

Compare Figures 1 and 3, which together illustrate Theorem 3.2 in the case of $G = SL_3$.

We point out that Conjecture 2 from [Bea09] about elements in the $v = w_0$ Weyl chamber follows as an immediate corollary of Theorem 3.2, since all minimal paths to w_0 in the quantum Bruhat graph consist exclusively of upward edges and thus carry no weight. On the other hand, this observation can also already be made as a direct consequence of Theorem 2.10, since in the antidominant chamber the translations are the minimal length coset representatives for \widetilde{W}/W .

The hypotheses of Theorem 3.2 depend on the pair $w, v \in W$ of finite Weyl group elements naturally associated to x . We remark, however, that property (2) of Proposition 3.1 provides a uniform bound on k for any finite Weyl group W . In particular, we know that $k \leq \ell(w_0)$, and so the uniform superregularity hypothesis $\langle \lambda, \alpha_i \rangle > 4\ell(w_0)$ for all $\alpha_i \in \Delta$ implies the stated hypothesis involving M_k for all classical groups. Of course, for any given pair $w, v \in W$, the minimum length of any path from $w^{-1}v$ to v might be considerably shorter than $\ell(w_0)$, and thus Theorem 3.2 places a strictly weaker superregularity hypothesis on λ . However, for the reader interested in a uniform statement for any $x \in \widetilde{W}$ in a fixed affine Weyl group, we make this observation formal in the following immediate corollary.

Corollary 3.3. *Let $x = t^{v\lambda}w \in \widetilde{W}$, and suppose that for all simple roots $\alpha_i \in \Delta$*

$$(3.2) \quad \langle \lambda, \alpha_i \rangle > \begin{cases} 4\ell(w_0) & \text{if } G \text{ is classical,} \\ 12\ell(w_0) & \text{if } G \text{ is exceptional.} \end{cases}$$

Then the maximum Newton point associated to x equals $\nu_x = \lambda - \alpha_x^\vee$, where α_x^\vee is the weight of any path of minimal length from $w^{-1}v$ to v in the quantum Bruhat graph for W .

We remark that one can also formally restate Corollaries 3.5 and 3.6 in the sections which follow using the same uniform superregularity hypothesis as in Corollary 3.3.

3.3. Connections to quantum Schubert calculus. We now discuss a surprising connection between the main result about Newton points, which arises from questions in the algebraic geometry of affine flag varieties in characteristic $p > 0$, to the quantum cohomology of standard complete flag varieties over \mathbb{C} . Given a complex reductive group G , the classical cohomology of the complete flag variety G/B over \mathbb{C} is a free \mathbb{Z} -module generated by Schubert classes, which are indexed by elements in the Weyl group W . If we define $\mathbb{Z}[q] := \mathbb{Z}[q_1, \dots, q_r]$, where r is the rank of G , the quantum cohomology ring of G/B then equals $QH^*(G/B) = H^*(G/B, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a $\mathbb{Z}[q]$ -module, and will also have a $\mathbb{Z}[q]$ -basis of Schubert classes σ_w where $w \in W$. The main problem in modern quantum Schubert calculus is to explicitly compute the products

$$(3.3) \quad \sigma_u * \sigma_v = \sum_{w,d} c_{u,v}^{w,d} q^d \sigma_w,$$

by finding non-recursive, positive combinatorial formulas for the *Gromov-Witten invariants* $c_{u,v}^{w,d}$ and the quantum parameters q^d . Roughly speaking, these Gromov-Witten invariants count the number of curves of *degree* d meeting a triple of Schubert varieties determined by $u, v, w \in W$.

Theorem 3.2 turns out to be related to the question of determining which degrees arise in the product of two Schubert classes. In [Pos05], Postnikov strengthens a result of [FW04], which both proves the existence of and then provides a combinatorial formula for the unique minimal monomial q^d which occurs with nonzero coefficient in any quantum Schubert product.

Theorem 3.4 (Corollary 3 [Pos05]). *Given any pair $u, v \in W$, the unique minimal monomial that occurs in the quantum product $\sigma_u * \sigma_v$ equals q^d , where d is the weight of any path of minimal length in the quantum Bruhat graph from u to w_0v .*

The following corollary relates this result in quantum Schubert calculus to our problem of finding maximal Newton points.

Corollary 3.5. *Fix any $u, v \in W$, and define k to be the length of any minimal path from u to w_0v in the quantum Bruhat graph. Let $\lambda \in Q^\vee$ be any coroot such that $\langle \lambda, \alpha_i \rangle > M_k$ for all $\alpha_i \in \Delta$. Then the following are equivalent:*

- (1) $q^d = q_1^{d_1} \cdots q_r^{d_r}$ is the minimal monomial in the quantum product $\sigma_u * \sigma_v$
- (2) $\lambda - d_1 \alpha_1^\vee - \cdots - d_r \alpha_r^\vee$ is the maximum Newton point in $\mathcal{N}(G)_x$, where $x = t^{w_0v(\lambda)} w_0v u^{-1}$.

Proof. By Theorem 3.4, q^d is the minimal monomial in the quantum product $\sigma_u * \sigma_v$ if and only if d is the weight of any path of minimal length in the quantum Bruhat graph from u to w_0v . Set $v' = w_0v$ and $w' = w_0v u^{-1}$, and compute that $(w')^{-1}v' = (w_0v u^{-1})(w_0v) = u$. Therefore, Theorem 3.2 says that the weight of any such path also gives the correction factor required to calculation ν_x for $x = t^{v'\lambda} w'$. \square

3.4. Affine Deligne-Lusztig varieties and Mazur's inequality. In [DL76], Deligne and Lusztig constructed a family of varieties X_w in $G(\overline{\mathbb{F}}_q)/B$ indexed by elements $w \in W$ to study the representation theory of finite Chevalley groups. Rapoport introduced *affine Deligne-Lusztig varieties* in [Rap00], defined as generalizations of these classical Deligne-Lusztig varieties. Although Deligne and Lusztig's original construction was motivated by applications to representation theory [Lus78], interest in affine Deligne-Lusztig varieties is rooted in their intimate relationship to reductions modulo p of Shimura varieties, among other arithmetic applications, many of which lie at the heart of the Langlands program; see [Rap05].

For $x \in \widetilde{W}$ and $b \in G(F)$, the associated affine Deligne-Lusztig variety is defined as

$$(3.4) \quad X_x(b) := \{g \in G(F)/I \mid g^{-1}b\sigma(g) \in IxI\}.$$

Unlike in the classical case in which Lang's Theorem automatically says that X_w is non-empty for every $w \in W$, affine Deligne-Lusztig varieties frequently tend to be empty. Providing a complete characterization for the pairs (x, b) for which the associated affine Deligne-Lusztig variety is non-empty has proven to be a surprisingly challenging problem. In the context of affine Deligne-Lusztig varieties inside the affine Grassmannian, the non-emptiness question is phrased in terms of *Mazur's inequality*, which relates the coroot λ from the translation part of x and the Newton point of b ; see [Maz72] and [Kat79]. If $\nu(b)$ denotes the Newton point associated to b , Mazur's inequality roughly states that $\nu(b) \leq \lambda^+$.

Although no simple analog of Mazur's inequality can perfectly predict whether or not $X_x(b)$ is non-empty, Theorem 3.2 yields an Iwahori analog of Mazur's inequality, providing a necessary condition for non-emptiness under a superregularity hypothesis on the coroot. The following corollary can be viewed as a refinement of Mazur's inequality on the affine Grassmannian for the the context of the affine flag variety.

Corollary 3.6. *Let $x = t^{v\lambda}w \in \widetilde{W}$, and consider any path of minimal length k from $w^{-1}v$ to v in the quantum Bruhat graph for W . Suppose that $\langle \lambda, \alpha_i \rangle > M_k$ for all simple roots $\alpha_i \in \Delta$. Fix $b \in G(F)$, and denote by $\nu(b)$ the Newton point for b . If $X_x(b)$ is non-empty, then*

$$(3.5) \quad \nu(b) \leq \lambda - \alpha_x^\vee,$$

where α_x^\vee is the weight of the chosen path from $w^{-1}v$ to v .

Proof. Denote by $[b]$ the σ -conjugacy class of b . By definition, if $X_x(b) \neq \emptyset$, then $[b] \cap IxI \neq \emptyset$ as well. For any $g \in [b] \cap IxI$, since the Newton point is a σ -conjugacy class invariant, we know that $\nu(g) = \nu(b)$. By maximality of ν_x and the fact that $g \in IxI$, we thus also have $\nu(b) \leq \nu_x$. Finally, recall that $\nu_x = \lambda - \alpha_x^\vee$ by Theorem 3.2. \square

4. THE QUANTUM BRUHAT GRAPH AND AFFINE BRUHAT ORDER

This section generalizes a result of Lam and Shimozono from [LS10] which we reformulate as Proposition 4.2, proving that covering relations in affine Bruhat order correspond to edges in the quantum Bruhat graph. In particular, if one is interested in singling out translations below a given $x \in \widetilde{W}$, then iterated application of this observation yields a correspondence stated in Proposition 4.5 between saturated chains in affine Bruhat order and paths in the quantum Bruhat graph. Proposition 4.5 is the first of two key propositions required for the proof of Theorem 3.2.

4.1. Edges and covering relations. For affine Weyl group elements whose translation part is superregular, we now discuss a correspondence between edges in the quantum Bruhat graph and covering relations in affine Bruhat order. This correspondence plays a central role in verifying the equivariant quantum Chevalley-Monk rule in the proof of the Peterson isomorphism in [LS10].

We start by recalling a standard length formula for affine Weyl group elements written in terms of the translation part and the pair of naturally associated finite Weyl group elements.

Lemma 4.1. *Let $\lambda \in Q^+$ be regular dominant, and let $x = t^{v\lambda}w \in \widetilde{W}$. Then*

$$(4.1) \quad \ell(x) = \ell(t^\lambda) - \ell(v^{-1}w) + \ell(v) = \ell(t^\lambda) - \ell(w^{-1}v) + \ell(v) = \langle \lambda, 2\rho \rangle - \ell(w^{-1}v) + \ell(v).$$

When we write $x > y$, we mean that x covers y in Bruhat order; i.e. both $x \geq y$ and $\ell(x) = \ell(y) + 1$, or equivalently we say that y is a *cocover* of x .

Proposition 4.2 (Reformulation of Proposition 4.4 [LS10]). *Let $x = t^{v\lambda}w \in \widetilde{W}$, and let $r_\beta = t^{v\alpha}r_{v\alpha}$ be the affine reflection such that $r_\beta x$ reflects x across the $H_{v\alpha, n}$ hyperplane. Further suppose that for all $\alpha_i \in \Delta$,*

$$(4.2) \quad \langle \lambda, \alpha_i \rangle \geq \begin{cases} 2\ell(w_0) & \text{if } G \neq G_2, \\ 3\ell(w_0) & \text{if } G = G_2. \end{cases}$$

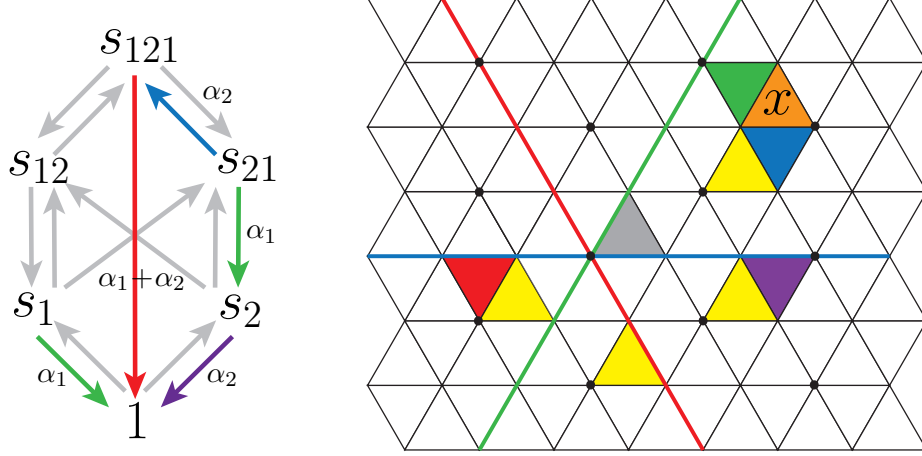


FIGURE 4. Cocovers of $x = t^{2\rho^\vee} s_{12}$ with corresponding edges in the quantum Bruhat graph drawn in the same color; maximal translations less than x are yellow.

Then $x \succ r_\beta x$ is a covering relation if and only if one of the following four conditions holds:

- (1) $\ell(vr_\alpha) = \ell(v) - 1$ and $n = 0$, in which case $r_\beta x = t^{vr_\alpha(\lambda)} r_{v\alpha} w$.
- (2) $\ell(vr_\alpha) = \ell(v) + \langle \alpha^\vee, 2\rho \rangle - 1$ and $n = 1$, in which case $r_\beta x = t^{vr_\alpha(\lambda - \alpha^\vee)} r_{v\alpha} w$.
- (3) $\ell(w^{-1}vr_\alpha) = \ell(w^{-1}v) + 1$ and $n = \langle \lambda, \alpha \rangle$, in which case $r_\beta x = t^{v(\lambda)} r_{v\alpha} w$.
- (4) $\ell(w^{-1}vr_\alpha) = \ell(w^{-1}v) - \langle \alpha^\vee, 2\rho \rangle + 1$ and $n = \langle \lambda, \alpha \rangle - 1$, in which case $r_\beta x = t^{v(\lambda - \alpha^\vee)} r_{v\alpha} w$.

Remark 4.3. Observe that each of the length conditions in the four cases of this proposition correspond to an edge in the quantum Bruhat graph. More precisely, writing the affine reflection $r_\beta = t^{nv\alpha^\vee} r_{v\alpha}$, we have the following association between the four cases in Proposition 4.2 and edges in the quantum Bruhat graph:

- (1) $x \succ r_\beta x$ corresponds to an upward edge into v of the form $vr_\alpha \rightarrow v$
- (2) $x \succ r_\beta x$ corresponds to a downward edge into v of the form $vr_\alpha \rightarrow v$
- (3) $x \succ r_\beta x$ corresponds to an upward edge out of $w^{-1}v$ of the form $w^{-1}v \rightarrow w^{-1}vr_\alpha$
- (4) $x \succ r_\beta x$ corresponds to a downward edge out of $w^{-1}v$ of the form $w^{-1}v \rightarrow w^{-1}vr_\alpha$

Example 4.4. Before proceeding with the proof, we provide an example which illustrates the correspondence established by Proposition 4.2 and Remark 4.3. Consider $x = t^{2\rho^\vee} s_{12}$, which is the orange alcove in Figure 4. In order to find all cocovers $r_\beta x \prec x$, we look at edges going *into* $v = 1$ and *out of* $w^{-1}v = s_{21}$ in the quantum Bruhat graph; these are the five colored edges in Figure 4. The three downward edges into 1 correspond to three cocovers of type (2), each coming from a different type of reflection, and the corresponding alcoves are colored green, purple, and red, respectively. There is one upward edge and one downward edge directed out of s_{21} , corresponding to cocovers of type (3) and (4), respectively. Note that two different edges give the same green cocover which shares a face with x —this situation can arise when x is close to the wall of a Weyl chamber.

Proposition 4.5 below then explains how each of the two minimal length *paths* from $w^{-1}v = s_{21}$ to $v = 1$ give rise to 8 different chains of length 2 from x to one of the yellow translation alcoves. Importantly, observe that these translations are based at coroots which lie in the same W -orbit.

Our proof of Proposition 4.2 closely follows the proof of Proposition 4.4 in [LS10], but we include all the details both in order to extract a precise superregularity hypothesis on λ , and also because we adopt several different conventions in this paper.

Proof of Proposition 4.2. First compute directly that

$$(4.3) \quad r_\beta x = t^{nv\alpha^\vee} r_{v\alpha} t^{v\lambda} w = t^{(nv\alpha^\vee + r_{v\alpha}v\lambda)} r_{v\alpha} w = t^{v(n\alpha^\vee + r_\alpha\lambda)} r_{v\alpha} w.$$

We can then rewrite this expression in one of two equivalent ways by either factoring out r_α or using the action of r_α on λ :

$$(4.4) \quad r_\beta x = t^{vr_\alpha(\lambda - n\alpha^\vee)} r_{v\alpha} w = t^{v(\lambda - (\langle\lambda, \alpha\rangle - n)\alpha^\vee)} r_{v\alpha} w.$$

Now recall that for any $\alpha_i \in \Delta$, we have by hypothesis

$$(4.5) \quad \langle\lambda, \alpha_i\rangle \geq \begin{cases} 2\ell(w_0) \geq \ell(w_0)\langle\alpha^\vee, \alpha_i\rangle & \text{if } G \neq G_2, \\ 3\ell(w_0) \geq \ell(w_0)\langle\alpha^\vee, \alpha_i\rangle & \text{if } G = G_2. \end{cases}$$

Here we have used the fact that the maximum value of $\langle\beta^\vee, \alpha\rangle$ in any reduced root system equals 2 in every Lie type, except for G_2 in which it equals 3; see [Bou02, Ch. VI §1, no. 3] Therefore, for any $\alpha_i \in \Delta$, we have

$$(4.6) \quad \langle\lambda - \ell(w_0)\alpha^\vee, \alpha_i\rangle \geq 0,$$

which shows that if n is any integer such that $n \leq \ell(w_0)$, then the difference $\lambda - n\alpha^\vee$ is dominant. Similarly, if $n \geq \langle\lambda, \alpha\rangle - \ell(w_0)$, then for any $\alpha_i \in \Delta$,

$$(4.7) \quad \langle\lambda, \alpha_i\rangle \geq \ell(w_0)\langle\alpha^\vee, \alpha_i\rangle \geq (\langle\lambda, \alpha\rangle - n)\langle\alpha^\vee, \alpha_i\rangle,$$

which means that

$$(4.8) \quad \langle\lambda - (\langle\lambda, \alpha\rangle - n)\alpha^\vee, \alpha_i\rangle \geq 0,$$

and so $\lambda - (\langle\lambda, \alpha\rangle - n)\alpha^\vee$ is dominant in this case.

Following [LS10], now define the convex function $f : \mathbb{Z} \rightarrow \mathbb{Z}_{\geq 0}$ by

$$(4.9) \quad f(n) = \ell(t^{v(\lambda - n\alpha^\vee)}) = \ell(t^{vr_\alpha(\lambda - n\alpha^\vee)}),$$

and note that $f(0) = \ell(t^{v\lambda}) = \langle\lambda, 2\rho\rangle = f(\langle\lambda, \alpha\rangle)$. The key idea here is that for integers n reasonably close to either 0 or $\langle\lambda, \alpha\rangle$, the function value $f(n)$ gives a good approximation for $\ell(x)$ and thus also for $\ell(r_\beta x) = \ell(x) - 1$. More precisely, first suppose that $\lambda - n\alpha^\vee$ is dominant. Then we can compute $f(n) = \ell(t^{vr_\alpha(\lambda - n\alpha^\vee)}) = \langle\lambda - n\alpha^\vee, 2\rho\rangle$ by Lemma 4.1. The difference between $\ell(r_\beta x)$ and $f(n)$ can therefore also be directly calculated:

$$(4.10) \quad \ell(r_\beta x) - f(n) = \ell(v) - \ell(w^{-1}v) - 1 + n\langle\alpha^\vee, 2\rho\rangle.$$

We can bound this difference below by observing that for any $w, v \in W$, we have $|\ell(v) - \ell(w^{-1}v)| \leq \ell(w_0)$, and $\langle\alpha^\vee, 2\rho\rangle \geq 2$ for any $\alpha \in R^+$. Therefore,

$$(4.11) \quad \ell(r_\beta x) - f(n) \geq -\ell(w_0) - 1 + 2n.$$

When $n = \ell(w_0)$, this lower bound is precisely $\ell(w_0) - 1$. On the other hand, since $t^{vr_\alpha(\lambda - n\alpha^\vee)}$ is the translation part of $r_\beta x$, then $|\ell(r_\beta x) - f(n)| \leq \ell(w_0)$. Recall that f is convex, and so f is decreasing as n increases from $n = 0$. Moreover, while f is decreasing, $f(m+1) - f(m) \geq 2$, where the lower bound occurs in the case of $\alpha \in \Delta$. Therefore, if $r_\beta x$ is a cocover of x and n is closer to 0, then necessarily $n \leq \ell(w_0)$. Similarly, if $\lambda - (\langle\lambda, \alpha\rangle - n)\alpha^\vee$ is dominant, then Lemma 4.1 says that $f(n) = \ell(t^{v(\lambda - (\langle\lambda, \alpha\rangle - n)\alpha^\vee)}) = \langle\lambda - (\langle\lambda, \alpha\rangle - n)\alpha^\vee, 2\rho\rangle$, in which case

$$(4.12) \quad \ell(r_\beta x) - f(n) = \ell(v) - \ell(w^{-1}v) - 1 + (\langle\lambda, \alpha\rangle - n)\langle\alpha^\vee, 2\rho\rangle \geq -\ell(w_0) - 1 + 2(\langle\lambda, \alpha\rangle - n).$$

Again by convexity, we must have $n \geq \langle\lambda, \alpha\rangle - \ell(w_0)$ in the case that n is closer to $\langle\lambda, \alpha\rangle$.

Altogether, what we have shown is that if $r_\beta x$ is a cocover of x , then we may safely assume that either $\lambda - n\alpha^\vee$ is dominant or $\lambda + (n - \langle\lambda, \alpha\rangle)\alpha^\vee$ is dominant. Since $x \geq r_\beta x$, then we can also automatically conclude that $0 \leq n \leq \langle\lambda, \alpha\rangle$ by length considerations. Thus the remainder of the proof naturally breaks up into two cases.

First suppose that $\lambda - n\alpha^\vee \in Q^+$. Applying the length formula in (4.1), we obtain

$$(4.13) \quad \ell(r_\beta x) = \ell(t^{(\lambda - n\alpha^\vee)}) - \ell((vr_\alpha)^{-1}r_{v\alpha}w) + \ell(vr_\alpha) = \langle \lambda - n\alpha^\vee, 2\rho \rangle - \ell(v^{-1}w) + \ell(vr_\alpha),$$

where we have used that $r_{v\alpha} = vr_\alpha v^{-1}$ for any $v \in W$. Now use Equation (4.1) to compare $\ell(x) = \langle \lambda, 2\rho \rangle - \ell(v^{-1}w) + \ell(v)$, and compute the difference

$$(4.14) \quad \ell(x) - \ell(r_\beta x) = n\langle \alpha^\vee, 2\rho \rangle + \ell(v) - \ell(vr_\alpha).$$

The element $r_\beta x$ is a cocover of x if and only if $\ell(x) - \ell(r_\beta x) = 1$, which we see by the equation above occurs if and only if

$$(4.15) \quad \ell(v) - \ell(vr_\alpha) = 1 - n\langle \alpha^\vee, 2\rho \rangle.$$

Now, the difference between $\ell(vr_\alpha)$ and $\ell(v)$ for any $v \in W$ is bounded above by $\ell(r_\alpha) \leq \langle \alpha^\vee, 2\rho \rangle - 1$. (We remark that if G is simply laced, then in fact $\ell(r_\alpha) = \langle \alpha^\vee, 2\rho \rangle - 1$.) Therefore,

$$(4.16) \quad -\langle \alpha^\vee, 2\rho \rangle + 1 \leq \ell(v) - \ell(vr_\alpha) = 1 - n\langle \alpha^\vee, 2\rho \rangle,$$

which happens if and only if

$$(4.17) \quad 0 \leq (1 - n)\langle \alpha^\vee, 2\rho \rangle.$$

Since $0 \leq n \leq \langle \lambda, \alpha \rangle$, then in particular n is a positive integer. In addition, since α is a positive root, we know that $\langle \alpha^\vee, 2\rho \rangle \geq 0$. This means that there are really only two choices for n , namely

$$(4.18) \quad n = \begin{cases} 0 & \text{and } \ell(v) - \ell(vr_\alpha) = 1 \\ 1 & \text{and } \ell(v) - \ell(vr_\alpha) = 1 - \langle \alpha^\vee, 2\rho \rangle. \end{cases}$$

Recall from Equation (4.4) that in this case we write $r_\beta x = t^{vr_\alpha(\lambda - n\alpha^\vee)}r_{v\alpha}w$, and substitute these two possible values for n to obtain cases (1) and (2) of the proposition.

Now suppose that $\lambda - (\langle \lambda, \alpha \rangle - n)\alpha^\vee \in Q^+$. Applying Equation (4.1) in this case, we see that

$$\ell(r_\beta x) = \ell(t^{(\lambda - (\langle \lambda, \alpha \rangle - n)\alpha^\vee)}) - \ell(v^{-1}r_{v\alpha}w) + \ell(v) = \langle \lambda - (\langle \lambda, \alpha \rangle - n)\alpha^\vee, 2\rho \rangle - \ell(r_\alpha v^{-1}w) + \ell(v).$$

We now compute the difference between $\ell(x)$ and the cocover $\ell(r_\beta x)$ to be

$$(4.19) \quad 1 = \ell(x) - \ell(r_\beta x) = \ell(r_\alpha v^{-1}w) - \ell(v^{-1}w) - \langle (n - \langle \lambda, \alpha \rangle)\alpha^\vee, 2\rho \rangle,$$

which we rearrange to obtain

$$(4.20) \quad \ell(r_\alpha v^{-1}w) - \ell(v^{-1}w) = 1 + \langle (n - \langle \lambda, \alpha \rangle)\alpha^\vee, 2\rho \rangle.$$

Again use that $|\ell(r_\alpha u) - \ell(u)| \leq \langle \alpha^\vee, 2\rho \rangle - 1$ for any $u \in W$ to see that

$$(4.21) \quad -\langle \alpha^\vee, 2\rho \rangle + 1 \leq \ell(r_\alpha v^{-1}w) - \ell(v^{-1}w) = 1 + \langle (n - \langle \lambda, \alpha \rangle)\alpha^\vee, 2\rho \rangle,$$

which happens if and only if

$$(4.22) \quad 0 \leq (n - \langle \lambda, \alpha \rangle + 1)\langle \alpha^\vee, 2\rho \rangle.$$

Since $0 \leq n \leq \langle \lambda, \alpha \rangle$, we know that $n - \langle \lambda, \alpha \rangle \leq 0$. In addition, since α is a positive root, there are really only two viable choices for n in this case, namely

$$(4.23) \quad n = \begin{cases} \langle \lambda, \alpha \rangle & \text{and } \ell(r_\alpha v^{-1}w) - \ell(v^{-1}w) = 1 \\ \langle \lambda, \alpha \rangle - 1 & \text{and } \ell(r_\alpha v^{-1}w) - \ell(v^{-1}w) = 1 - \langle \alpha^\vee, 2\rho \rangle. \end{cases}$$

Observe that $\ell(r_\alpha v^{-1}w) - \ell(v^{-1}w) = \ell(w^{-1}vr_\alpha) - \ell(w^{-1}v)$, recall from Equation (4.4) that in this second case we write $r_\beta x = t^{v(\lambda + (n - \langle \lambda, \alpha \rangle)\alpha^\vee)}r_{v\alpha}w$, and substitute the two possible values for n from above to obtain cases (3) and (4) of the proposition. \square

4.2. Paths and saturated chains. As we will see in Proposition 6.1, in order to find the maximum Newton point ν_x , we will need to look for saturated chains in Bruhat order from x which terminate at a pure translation element. Repeated application of Proposition 4.2 provides an interpretation of such chains in terms of paths in the quantum Bruhat graph, and vice versa. Although we shall not use the full strength of the correspondence as stated in Proposition 4.5 in the proof of Theorem 3.2, we prove the most precise statement possible in case this proposition might be of independent combinatorial interest.

Proposition 4.5. *Let $w, v \in W$, and let k be the minimum length of any path in the quantum Bruhat graph from $w^{-1}v$ to v . Define $x = t^{v\lambda}w \in \widetilde{W}$, and suppose that*

$$(4.24) \quad \langle \lambda, \alpha_i \rangle \geq \begin{cases} 2\ell(w_0) + 2k - 2 & \text{if } G \neq G_2, \\ 3\ell(w_0) + 3k - 3 & \text{if } G = G_2. \end{cases}$$

for all $\alpha_i \in \Delta$. Then,

- (i) any minimal length saturated chain from x to a pure translation is of length k , and can be associated to a unique path in the quantum Bruhat graph from $w^{-1}v$ to v of length k .
- (ii) if $\ell(x) > k$, then any path of length k in the quantum Bruhat graph from $w^{-1}v$ to v can be lifted to 2^k saturated chains in Bruhat order of length k from x to a pure translation.

Proof. Both parts of this proof proceed by induction on k .

(i) First suppose that $k = 0$. In this case, $w^{-1}v = v$, which means that $w = 1$ and so $x = t^{v\lambda}$ is already a translation. We can thus associate to x the path of length 0 in the quantum Bruhat graph from v to itself.

Now suppose that $k \geq 1$, and consider a saturated chain of minimal length from x to a translation, say $x \succ x_1 \succ \cdots \succ x_m = t^\mu$. Using Proposition 4.2, we can explicitly write $x_1 = t^{v'\lambda'}r_{v\alpha}w$ for some $\alpha \in R^+$. There are several cases to consider for v' and λ' , depending on the type of the covering relation $x \succ x_1$. In any of the four cases in Proposition 4.2, either $\lambda' = \lambda$ or $\lambda' = \lambda - \alpha^\vee \in Q^+$. We can thus compute that for any $\alpha_i \in \Delta$, we have

$$(4.25) \quad \langle \lambda', \alpha_i \rangle \geq \langle \lambda - \alpha^\vee, \alpha_i \rangle \geq \begin{cases} (2\ell(w_0) + 2k - 2) - 2 = 2\ell(w_0) + 2(k - 1) - 2 & \text{if } G \neq G_2, \\ (3\ell(w_0) + 3k - 3) - 3 = 3\ell(w_0) + 3(k - 1) - 3 & \text{if } G = G_2. \end{cases}$$

For brevity, denote the finite part of x_1 by $w' = r_{v\alpha}w$. Recall that either $v' = vr_\alpha$ in cases (1) and (2) of Proposition 4.2, or $v' = v$ in cases (3) or (4). Compute using $r_{v\alpha} = vr_\alpha v^{-1}$ that

$$(4.26) \quad (w')^{-1}v' = \begin{cases} w^{-1}v, & \text{if } v' = vr_\alpha \\ w^{-1}vr_\alpha, & \text{if } v' = v. \end{cases}$$

Suppose we are in cases (1) or (2) so that $v' = vr_\alpha$ and $vr_\alpha \rightarrow v$ is an edge in the quantum Bruhat graph. Since any path from $(w')^{-1}v' = w^{-1}v$ to $v' = vr_\alpha$ can be extended by a single edge to a path from $w^{-1}v$ to v , then the minimal length of such a path must equal $k - 1$. We have thus verified that the induction hypothesis always applies to the element $x_1 \in \widetilde{W}$ and the pair $w', v' \in W$ in cases (1) and (2) of Proposition 4.2. Similarly, in cases (3) or (4) in which $v' = v$, the edge $w^{-1}v \rightarrow w^{-1}vr_\alpha$ is an edge in the quantum Bruhat graph. Any path from $(w')^{-1}v' = w^{-1}vr_\alpha$ to $v' = v$ can thus be extended by this edge to a path from $w^{-1}v$ to v , which means that the minimum length of a path from $(w')^{-1}v'$ to v' also equals $k - 1$ in this case. Therefore, the induction hypothesis still applies to x_1 and the pair $w', v' \in W$ in cases (3) and (4) of Proposition 4.2.

Therefore, independent of the type of the cocover x_1 , the induction hypothesis says that the path $(w')^{-1}v' \rightarrow \cdots \rightarrow v'$ of length $k - 1$ in the quantum Bruhat graph is associated to a unique saturated chain of the form $x_1 \succ x_2 \succ \cdots \succ x_k = t^\mu$ of length $k - 1$. The inductive hypothesis thus implies that $m \leq k$, but we cannot have $m < k$ since the argument applies to all cocovers x_1 of x at once. Therefore, in fact $m = k$.

To complete the proof of the inductive step we again consider two cases. First suppose that $v' = vr_\alpha$, and recall from Remark 4.3 that in cases (1) and (2) the covering relation $x \succ x_1$ corresponds to an edge of the form $vr_\alpha \rightarrow v$. Putting the path from $w^{-1}v$ to vr_α together with this final edge yields a single path of length k in the quantum Bruhat graph from $w^{-1}v$ to v . Similarly, if $v' = v$, then we are in case (3) or (4) of Remark 4.3, and the covering relation $x \succ x_1$ corresponds to an edge $w^{-1}v \rightarrow w^{-1}vr_\alpha$. Appending this edge to the beginning of the path from $w^{-1}vr_\alpha$ to v again yields a single path from $w^{-1}v$ to v . The statement (i) now follows by induction.

(ii) We again proceed by induction on k , noting that there is nothing to prove in the $k = 0$ case, which corresponds to the situation in which x itself is already a translation. Consider any path of length $k \geq 1$ from $w^{-1}v$ to v in the quantum Bruhat graph, say

$$(4.27) \quad w^{-1}v \rightarrow w^{-1}vr_{\beta_1} \rightarrow \cdots \rightarrow w^{-1}vr_{\beta_1} \cdots r_{\beta_{k-1}} \rightarrow w^{-1}vr_{\beta_1} \cdots r_{\beta_k} = v.$$

First apply cases (3) and (4) of Proposition 4.2 and Remark 4.3 to the initial edge of this path $w^{-1}v \rightarrow w^{-1}vr_{\beta_1}$. In either case we obtain a unique cocover $x_1 = t^{v\lambda'} r_{v\beta_1} w$ of x , where $\lambda' = \lambda$ in case (3) and $\lambda' = \lambda - \beta_1^\vee$ in case (4), corresponding to whether the edge $w^{-1}v \rightarrow w^{-1}vr_{\beta_1}$ is directed upward or downward in the graph, respectively. Now define $w' = r_{v\beta_1} w$, which is the finite part of x_1 , and compute that

$$(4.28) \quad (w')^{-1}v = (r_{v\beta_1} w)^{-1}v = w^{-1}(vr_{\beta_1} v^{-1})v = w^{-1}vr_{\beta_1}.$$

Note that since k is the minimum length of any path from $w^{-1}v$ to v , then the truncated path

$$(4.29) \quad (w')^{-1}v = w^{-1}vr_{\beta_1} \rightarrow \cdots \rightarrow v$$

of length $k - 1$ is a path of minimal length from $(w')^{-1}v$ to v in the quantum Bruhat graph. Of course, since $x_1 \prec x$ is a cocover, then since $\ell(x) > k$ by hypothesis, we know that $\ell(x_1) > k - 1$. Finally, recall Equation (4.25) to complete the verification of the induction hypothesis on the pair $w', v \in W$ and the element $x_1 = t^{v\lambda'} w' \in \widetilde{W}$. Therefore, by induction we obtain 2^{k-1} distinct saturated chains of length $k - 1$ from x_1 to a pure translation. To each of these chains, we can pre-append the covering relation $x \succ x_1$ to obtain 2^{k-1} saturated chains from x to a pure translation, all of which are length k .

We now consider the path in (4.27) from a different perspective. Looking at the final edge in this path which terminates at v , we see that $w^{-1}vr_{\beta_1} \cdots r_{\beta_{k-1}} = vr_{\beta_k}$. It will thus be more convenient to write this final edge as $vr_{\beta_k} \rightarrow v$. Now apply cases (1) and (2) of Proposition 4.2 and Remark 4.3 to this final edge. In either case, we obtain a unique cocover $y_1 = t^{v\beta_k(\mu)} r_{v\beta_k} w$ of x , where $\mu = \lambda$ in case (1) and $\mu = \lambda - \beta_k^\vee$ in case (2), corresponding to whether the edge $vr_{\beta_k} \rightarrow v$ is directed upward or downward in the graph, respectively. Now define $v' = vr_{\beta_k}$, which indexes the Weyl chamber in which the cocover y_1 lies, and denote by $w'' = r_{v\beta_k} w$, which is the finite part of y_1 . Compute that

$$(4.30) \quad (w'')^{-1}v' = (r_{v\beta_k} w)^{-1}vr_{\beta_k} = w^{-1}(vr_{\beta_k} v^{-1})vr_{\beta_k} = w^{-1}v.$$

By the same calculation as in (4.25), we know that

$$(4.31) \quad \langle \mu, \alpha_i \rangle \geq \begin{cases} 2\ell(w_0) + 2(k-1) - 2 & \text{if } G \neq G_2, \\ 3\ell(w_0) + 3(k-1) - 3 & \text{if } G = G_2. \end{cases}$$

Finally, the relationship $y_1 \prec x$ implies that $\ell(y_1) \geq k - 1$. Therefore, the induction hypothesis also applies to the pair $w'', v' \in W$, the element $y_1 \in \widetilde{W}$, and the truncated path

$$(4.32) \quad (w'')^{-1}v' = w^{-1}v \rightarrow \cdots \rightarrow vr_{\beta_k} = v'$$

of length $k - 1$ in the quantum Bruhat graph. We thus obtain 2^{k-1} distinct saturated chains of length $k - 1$ from y_1 to a pure translation, which can be concatenated with the covering $x \succ y_1$ to give 2^{k-1} saturated chains from x to a pure translation of length k .

Finally, we argue that under the superregularity hypothesis on λ and the additional hypothesis that $\ell(x) > 1$, then $x_1 \neq y_1$. Suppose for a contradiction that $x_1 = y_1$. We then have that

$$(4.33) \quad x_1 = t^{v(\lambda - m\beta_1^\vee)} r_{v\beta_1} w = t^{vr\beta_k(\lambda - n\beta_k^\vee)} r_{v\beta_k} w = y_1,$$

where $m, n \in \{0, 1\}$. Comparing the finite parts, we first see that $\beta_1 = \beta_k$, and so in the remainder of the argument we omit subscripts for convenience. Setting the translation parts equal to each other then says that

$$(4.34) \quad v(\lambda - m\beta^\vee) = vr_\beta(\lambda - n\beta^\vee) \iff$$

$$(4.35) \quad v\lambda - mv\beta^\vee = v\lambda - \langle \lambda, \beta \rangle v\beta^\vee + nv\beta^\vee \iff$$

$$(4.36) \quad \vec{0} = (m + n - \langle \lambda, \beta \rangle) v\beta^\vee \iff$$

$$(4.37) \quad \langle \lambda, \beta \rangle = m + n.$$

On the other hand, since $m, n \in \{0, 1\}$, then $m + n \leq 2$. Comparing this inequality to our superregularity hypothesis (4.24), we deduce that $G = A_1$ and $\beta = \alpha_1$ so that we can attain the minimum possible value of $\langle \lambda, \beta \rangle = 2$. A direct calculation shows that there are only two possible elements $x \in \widetilde{S}_2$ which satisfy these criteria. Namely, either $x = s_0 = t^{\alpha_1} s_1$, which is excluded by the hypothesis that $\ell(x) > k = 1$ (and for which the statement of the proposition fails, since in this case there is a single cocover of x rather than two as the proposition predicts), or $x = t^{s_1 \alpha_1} s_1$. One can compute directly that there are indeed two distinct cocovers of $x = t^{s_1 \alpha_1} s_1$ coming from Proposition 4.2, namely $x_1 = t^{s_1 \alpha_1}$ and $y_1 = t^{\alpha_1}$. Therefore, in any case permitted by our hypotheses, the cocovers x_1 and y_1 are distinct.

Since $x_1 \neq y_1$, then each of the chains of the form $x \succ x_1 \succ \cdots \succ t^\nu$ is distinct from each of the chains of the form $x \succ y_1 \succ \cdots \succ t^\gamma$. Altogether we thus have $2^{k-1} + 2^{k-1} = 2^k$ saturated chains of length k from x to a pure translation, and so the result follows by induction. \square

Remark 4.6. We remark that the length hypothesis $\ell(x) > k$ in part (ii) arises specifically to exclude the situation in which $x_{k-1} = s_0$ in the saturated chain, in which case there is a single cocover $x_k = 1$, rather than two. If one wishes to drop this length hypothesis, the argument applies all the same, but the count on saturated chains equals 2^{k-1} in this case, rather than 2^k as stated. As we will see in Sublemma 5.11, however, the superregularity hypothesis on λ in Theorem 3.2 actually implies this length hypothesis on x , and so it is a harmless addition.

5. CONVEXITY, DOMINANCE ORDER, AND ROOT HYPERPLANES

This section lays the necessary technical groundwork for the proof of our second key proposition in Section 6.1. The essential idea is to construct upper and lower bounds on the maximum Newton point ν_x using the geometry of the affine hyperplane arrangement to compare Newton points in dominance order. For example, Lemma 5.3 constructs an element which bounds the Newton point $\nu(x)$ for x from above. We also highlight Lemma 5.7 defining a linear functional which we then use to bound ν_x from below in Lemma 5.12.

5.1. Convexity and dominance order. One key ingredient in the proof of the main theorem uses a geometric interpretation of the dominance order on the poset $Q^+ \otimes_{\mathbb{Z}} \mathbb{Q}$ in terms of convex subsets of Euclidean space. We thus state the following lemma due to Atiyah and Bott, which they also attribute to earlier work of Horn [Hor54] and Kostant [Kos73].

Lemma 5.1 (Lemma 12.14 [AB83]). *Let $x, y \in C$ be any points in the closed dominant Weyl chamber. Then*

$$(5.1) \quad y \leq x \iff y \in \text{Conv}(Wx),$$

where $\text{Conv}(Wx)$ denotes the convex hull of the W -orbit of the point x .

We remark that this statement also holds when x and y are elements of the integral weight lattice; see Theorem 1.9 in [Ste98]. In fact, it is possible to adapt Stembridge's proofs to the case of points in $Q^+ \otimes_{\mathbb{Z}} \mathbb{Q}$ with bounded denominator, which is sufficient for our purposes given that all Newton points with rational slopes satisfy an integrality condition. We nevertheless appeal to the result from [AB83] since it holds for any vectors in \mathbb{R}^r in the dominant Weyl chamber.

5.2. Variations on Mazur's inequality. Given an element $x = t^\lambda w$ in the affine Weyl group, Proposition 2.5 and Lemma 5.1 show that

$$(5.2) \quad \nu(x) \leq \nu(t^\lambda) = \lambda^+$$

Alternatively, this observation can be thought of as an application of Mazur's inequality. Generically, however, the Newton point of an affine Weyl group element whose finite part is non-trivial will lie *strictly* below the Newton point of the corresponding pure translation element. We state this slight strengthening of Mazur's inequality in the following simple lemma.

Lemma 5.2. *If $x = t^\lambda w \in \widetilde{W}$ is such that λ is regular and $w \neq 1$, then $\nu(x) < \nu(t^\lambda)$.*

Proof. Observe by Proposition 2.5 that $\nu(t^\lambda) = \lambda^+$. Proposition 2.5 also says that in general the Newton point $\nu(x)$ for the element x is given by averaging the orbit of λ under the powers of w . Therefore, $\nu(x) = \lambda^+$ if and only if $w(\lambda) = \lambda$; i.e. this occurs when w is in the stabilizer in W of λ . However, a basic fact from [Bou02] says that $\text{Stab}_W(\lambda)$ is generated by the reflections in W that fix λ . Therefore, $\text{Stab}_W(\lambda)$ is trivial as long as λ is regular and does not lie on any root hyperplane. \square

Generally speaking, if $x = t^\lambda w$ and the order of $w \in W$ is large, the Newton point $\nu(x)$ is actually considerably smaller than λ^+ . When the finite part of x is a simple reflection, however, the Newton point of x is as close as possible to λ^+ . The next lemma shows that this extreme case in which $w = s_i$ interpolates between the prediction of Mazur's inequality and the actual Newton point $\nu(x)$.

Lemma 5.3. *Let $x = t^\lambda w \in \widetilde{W}$ with $\lambda \in Q^+$ dominant and $w \neq 1$. Then there exists a simple reflection $s_i \in W$ such that $\lambda \geq \nu(t^\lambda s_i) \geq \nu(x)$.*

Proof. By the fact that λ is dominant and by Mazur's inequality (5.2), we automatically have that $\lambda \geq \nu(x)$. In addition, for any choice of $s \in S$, we also get $\lambda \geq \nu(t^\lambda s)$ again by dominance and Lemma 5.2. It therefore remains to choose a suitable $s_i \in S$ such that $\nu(t^\lambda s_i) \geq \nu(x)$.

Since $w \neq 1$, then the order of w equals $m \geq 2$. Clearly, if we already have $x = t^\lambda s_i$, then there is nothing more to do. More generally, recall from Theorem 2.5 that $\nu(x) = \left(\frac{1}{m} \sum_{i=1}^m w^i(\lambda) \right)^+$.

Denote by $o(x) = \frac{1}{m} \sum_{i=1}^m w^i(\lambda)$, which equals the average of the orbit of w on λ . Geometrically, $o(x)$ is the barycenter of the convex hull of the set of points $\{\lambda, w(\lambda), \dots, w^{m-1}(\lambda)\}$. Note that $o(x)$ lies on the wall of some (not necessarily dominant) Weyl chamber, and that $\nu(x)$ then equals the unique dominant element in the W -orbit of $o(x)$. Therefore, $\nu(x)$ lies on a wall of the dominant Weyl chamber (perhaps even the intersection of several walls). Choose any $1 \leq i \leq n$ such that $\nu(x) \in H_{\alpha_i}$, and define $y = t^\lambda s_i$ so that $\nu(y) = \lambda - \frac{\langle \lambda, \alpha_i \rangle}{2} \alpha_i^\vee$.

We claim that $\nu(y) \geq \nu(x)$. Recall that $\lambda \geq \nu(x)$, which means by definition that $\lambda - \nu(x) = \sum_{i=1}^n r_i \alpha_i^\vee$ for some nonnegative rational numbers $r_i \in \mathbb{Q}_{\geq 0}$. Of course, since $\lambda \in Q^+$ and $\lambda - \nu(y) = \frac{\langle \lambda, \alpha_i \rangle}{2} \alpha_i^\vee$, then it is also clear that $\lambda \geq \nu(y)$. Taking differences, we see that $\nu(y) - \nu(x) = \sum_{j \neq i} r_j \alpha_j^\vee + \left(r_i - \frac{\langle \lambda, \alpha_i \rangle}{2} \right) \alpha_i^\vee$. Note, however, that both $\nu(x)$ and $\nu(y)$ are in the hyperplane H_{α_i} . Denote by

$\text{proj}_i : \mathbb{R}^r \longrightarrow H_{\alpha_i}$ the orthogonal projection onto this hyperplane. We can then equivalently express the difference $\nu(y) - \nu(x) = \text{proj}_i \left(\sum_{j \neq i} r_j \alpha_j^\vee \right) = \sum_{j \neq i} r_j \text{proj}_i \left(\alpha_j^\vee \right)$.

We directly compute that $\text{proj}_i(\alpha_j^\vee) = \alpha_j^\vee - \frac{\langle \alpha_j^\vee, \alpha_i \rangle}{2} \alpha_i^\vee$. Since each column of the Cartan matrix contains a unique positive entry on the diagonal, whenever $i \neq j$ we know that $\langle \alpha_j^\vee, \alpha_i \rangle \leq 0$; see [Bou02, Ch. VI §1, no. 5]. Therefore, each vector $\text{proj}_i(\alpha_j^\vee)$ is a nonnegative rational sum of coroots. Since each $r_j \in \mathbb{Q}_{\geq 0}$, we have thus shown that $\nu(y) \geq \nu(x)$ in dominance order. \square

Notation 5.4. Given $x = t^\lambda w \in \widetilde{W}$ with $\lambda \in Q^+$, we shall denote by $\nu_i(x)$ the Newton point of the element $t^\lambda s_i$ whose existence is guaranteed by Lemma 5.3. The element $\nu_i(x)$ can be thought of as the maximal element in H_{α_i} such that $\lambda \geq \nu_i(x) \geq \nu(x)$, although we shall not use this property in its full strength.

Several of the lemmas in this section require that x lie in the dominant Weyl chamber, and so the next lemma makes a necessary reduction to the dominant case for the sake of calculating the Newton point of x .

Lemma 5.5. *Let $x = t^{v\lambda} w \in \widetilde{W}$, where $\lambda \in Q^+$ is dominant and $v, w \in W$. Then*

$$(5.3) \quad \nu(x) = \nu(t^\lambda v^{-1} w v).$$

Proof. Suppose that the order of $w \in W$ equals m , and consider the following expression:

$$(5.4) \quad \frac{1}{m} \sum_{i=1}^m w^i(v\lambda) = \frac{1}{m} \sum_{i=1}^m v v^{-1} w^i(v\lambda) = v \left(\frac{1}{m} \sum_{i=1}^m (v^{-1} w v)^i(\lambda) \right).$$

Since we know from Proposition 2.5 that $\nu(x) = \left(\frac{1}{m} \sum_{i=1}^m w^i(v\lambda) \right)^+$, then the expression on the right in (5.4) has the same Newton point as x . That is, since we take the unique dominant element in the W -orbit of this expression, applying v has no effect on the calculation of the Newton point. Therefore, the element in \widetilde{W} with translation part $\lambda \in Q^+$ and finite part equal to $v^{-1} w v$ has the same Newton point as x . \square

5.3. Root hyperplanes and convexity. In order to apply Lemma 5.1 to compare pairs of Newton points, we need a criterion which easily determines whether or not a point in \mathbb{R}^r lies in the convex hull of the W -orbit of another. In light of Lemma 5.3, we are particularly interested in the W -orbit of points of the form $\nu_i(x)$ which lie on some wall H_{α_i} of the dominant Weyl chamber. Lemma 5.7 below defines a linear functional which determines the hyperplane containing a codimension one face of $\text{Conv}(W\nu_i(x))$.

Definition 5.6. Define a hyperplane $H_{\widehat{\alpha}_i}$ in \mathbb{R}^r which has basis $\Delta \setminus \{\alpha_i\}$ for some $i \in \{1, \dots, n\}$.

The hyperplanes $H_{\widehat{\alpha}_i}$ also appear in [Hit10] where they are called *dual hyperplanes*; the terminology comes from the fact that the hyperplane $H_{\widehat{\alpha}_i}$ is orthogonal to the fundamental weight ω_i .

Lemma 5.7. *The hyperplane $H_{\widehat{\alpha}_i}$ is determined by the linear functional $F_i : \mathbb{R}^r \longrightarrow \mathbb{R}$ given by*

$$(5.5) \quad F_i(\vec{v}) := \left\langle \vec{v}, \sum_{\beta \in R^+ \setminus R_P^+} \beta \right\rangle = 0,$$

where P is the maximal parabolic subgroup defined by $\Delta_P = \Delta \setminus \{\alpha_i\}$ and R_P^+ is the corresponding set of positive roots. In other words, the hyperplane $H_{\widehat{\alpha}_i}$ is normal to the vector $\sum \beta$.

We remark that Lemma 5.7 does not appear to be well-known in the literature, and so the author would be grateful to any reader who could provide a reference for the characterization of the normal vector appearing in this lemma.

Proof. Denote by (\cdot, \cdot) the usual Euclidean inner product. We first review several equalities on the Euclidean inner product of certain roots. For any simple root $\alpha_j \in \Delta$ in a reduced root system, Proposition 29 in [Bou02, Ch. VI §1, no. 10] says that

$$(5.6) \quad (\alpha_j, 2\rho) = (\alpha_j, \alpha_j).$$

Therefore, we can directly compute using Equation (5.6) that for any $\alpha_j \in \Delta$, we have

$$(5.7) \quad (\alpha_j^\vee, 2\rho) = \left(\frac{2\alpha_j}{(\alpha_j, \alpha_j)}, 2\rho \right) = \frac{2}{(\alpha_j, \alpha_j)} (\alpha_j, 2\rho) = 2.$$

Now fix any $1 \leq i \leq n$ and consider the hyperplane $H_{\widehat{\alpha}_i}$ which has basis $\Delta_P = \Delta \setminus \{\alpha_i\}$ given by all of the other simple roots. Since $H_{\widehat{\alpha}_i}$ is a hyperplane in \mathbb{R}^r , the vectors in $H_{\widehat{\alpha}_i}$ will be determined by a single normal vector. We claim that the vector $\sum_{\beta \in R^+ \setminus R_P^+} \beta$ is normal to every element of Δ_P

and is therefore a normal vector to $H_{\widehat{\alpha}_i}$. To this end, fix any $1 \leq k \leq n$ such that $i \neq k$. Partition the positive roots into the two sets $R^+ = R_P^+ \sqcup (R^+ \setminus R_P^+)$, and compute using linearity that

$$(5.8) \quad 2 = (\alpha_k^\vee, 2\rho) = (\alpha_k^\vee, \sum_{\alpha \in R^+} \alpha) = \left(\alpha_k^\vee, \sum_{\beta \in R^+ \setminus R_P^+} \beta \right) + \left(\alpha_k^\vee, \sum_{\gamma \in R_P^+} \gamma \right).$$

Rewrite the right summand as

$$(5.9) \quad \left(\alpha_k^\vee, \sum_{\gamma \in R_P^+} \gamma \right) = (\alpha_k^\vee, 2\rho_P),$$

where ρ_P is the half-sum of all of the positive roots corresponding to the parabolic subgroup P . Although the root system R_P might be reducible and perhaps has a different Lie type than R , since R_P is still reduced, Equation (5.7) says that $(\alpha_k^\vee, 2\rho_P) = 2$. Putting this observation together with Equation (5.8), we have

$$(5.10) \quad 2 = \left(\alpha_k^\vee, \sum_{\beta \in R^+ \setminus R_P^+} \beta \right) + 2,$$

which means that $\left(\alpha_k^\vee, \sum_{\beta \in R^+ \setminus R_P^+} \beta \right) = 0$, and the result follows. \square

In order to determine whether a point in $Q^+ \otimes_{\mathbb{Z}} \mathbb{Q}$ lies on one side or the other of the hyperplane $H_{\widehat{\alpha}_i}$, we require the following positivity lemma.

Lemma 5.8. *For any $\alpha \in R^+$ and any $1 \leq i \leq n$, we have $F_i(\alpha^\vee) \geq 0$.*

Proof. First note that since $F_i : \mathbb{R}^r \rightarrow \mathbb{R}$ is a linear functional, then it suffices to prove the statement for any simple coroot α_j^\vee . By construction, if $i \neq j$ we know that $F_i(\alpha_j^\vee) = 0$, and so it remains

G	expression for the highest coroot
A_n	$\alpha_1^\vee + \cdots + \alpha_n^\vee$
B_n	$2\alpha_1^\vee + \cdots + 2\alpha_{n-1}^\vee + \alpha_n^\vee$
C_n	$\alpha_1^\vee + 2\alpha_2^\vee + \cdots + 2\alpha_n^\vee$
D_n	$\alpha_1^\vee + 2\alpha_2^\vee + \cdots + 2\alpha_{n-2}^\vee + \alpha_{n-1}^\vee + \alpha_n^\vee$
E_6	$\alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee + 3\alpha_4^\vee + 2\alpha_5^\vee + \alpha_6^\vee$
E_7	$2\alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee + 4\alpha_4^\vee + 3\alpha_5^\vee + 2\alpha_6^\vee + \alpha_7^\vee$
E_8	$2\alpha_1^\vee + 3\alpha_2^\vee + 4\alpha_3^\vee + 6\alpha_4^\vee + 5\alpha_5^\vee + 4\alpha_6^\vee + 3\alpha_7^\vee + 2\alpha_8^\vee$
F_4	$2\alpha_1^\vee + 4\alpha_2^\vee + 3\alpha_3^\vee + 2\alpha_4^\vee$
G_2	$2\alpha_1^\vee + 3\alpha_2^\vee$

TABLE 1. The highest coroot $\tilde{\mu}$ satisfies $\tilde{\mu} \geq \beta^\vee$ for all $\beta^\vee \in R^\vee$.

only to prove the positivity of $F_i(\alpha_i^\vee)$.

$$(5.11) \quad F_i(\alpha_i^\vee) = \left\langle \alpha_i^\vee, \sum_{\beta \in R^+ \setminus R_P^+} \beta \right\rangle = \frac{2 \left(\alpha_i^\vee, \sum_{\beta \in R^+ \setminus R_P^+} \beta \right)}{\left(\sum_{\beta \in R^+ \setminus R_P^+} \beta, \sum_{\beta \in R^+ \setminus R_P^+} \beta \right)},$$

from which we see that $F_i(\alpha_i^\vee) \geq 0$ if and only if $\left(\alpha_i^\vee, \sum_{\beta \in R^+ \setminus R_P^+} \beta \right) \geq 0$. Partitioning the roots as in Equation (5.8), we can write

$$(5.12) \quad 2 = (\alpha_i^\vee, 2\rho) = \left(\alpha_i^\vee, \sum_{\beta \in R^+ \setminus R_P^+} \beta \right) + \left(\alpha_i^\vee, \sum_{\gamma \in R_P^+} \gamma \right).$$

Recall that R_P^+ is spanned by $\Delta_P = \Delta \setminus \{\alpha_i\}$. Because each column in every Cartan matrix has a unique positive entry on the diagonal, we know that $\langle \alpha_i^\vee, \alpha_k \rangle \leq 0$ and therefore $(\alpha_i^\vee, \alpha_k) \leq 0$ for all $i \neq k$. Therefore, by linearity we know that the sum $\left(\alpha_i^\vee, \sum_{\gamma \in R_P^+} \gamma \right) \leq 0$. Comparing this observation to the positive sum in Equation (5.12), we must have $F_i(\alpha_i^\vee) \geq 0$, as desired. \square

5.4. Compatibility of superregularity hypotheses. Our final lemma requires a slight variation on our previous superregular hypothesis on λ , and the new ingredient arises from decomposing the highest coroot in R^\vee in terms of the simple coroots. Denote the highest *root* by θ (also often denoted in the literature by $\tilde{\alpha}$), which has the property that $\beta \leq \theta$ for any $\beta \in R$. Since R^\vee is also a root system, there is similarly a coroot which is the maximum element in R^\vee with respect to dominance order. We call this coroot the *highest coroot*, and denote it by $\tilde{\mu}$.

We provide a table of values for $\tilde{\mu}$ for the reader's convenience in Table 1, although this table can easily be obtained from the corresponding information on the highest root by duality; compare Table 4.1 in [Hum90]. (Note that $\tilde{\mu} \neq \theta^\vee$ unless G is simply laced; *i.e.* $\tilde{\mu} = \theta^\vee$ if and only if G is of type A, D or E .) We follow the conventions in [Bou02] for labeling the simple (co)roots, which are consistent with the conventions in [Hum90].

We now extract the essential information from Table 1 for the purpose of our final lemma.

Definition 5.9. Define a constant $1 \leq c_G \leq 6$ to be the maximum integer required to express the highest coroot in the basis of simple coroots. More specifically, define

$$(5.13) \quad c_G = \begin{cases} 1 & \text{if } G = A_n, \\ 2 & \text{if } G = B_n, C_n, \text{ or } D_n, \\ 3 & \text{if } G = E_6 \text{ or } G_2, \\ 4 & \text{if } G = E_7 \text{ or } F_4, \\ 6 & \text{if } G = E_8. \end{cases}$$

Before proceeding with the proof of our final lemma, we define a single constant which unifies the various required bounds on $\langle \lambda, \alpha_j \rangle$ from previous sections with the superregularity hypothesis necessary for Lemma 5.12 below.

Definition 5.10. Given a pair $w, v \in W$ of finite Weyl group elements, suppose that k is the minimum length of any path from $w^{-1}v$ to v in the quantum Bruhat graph for W . Define

$$(5.14) \quad M_k = \begin{cases} \max\{2\ell(w_0) + 2k - 2, 2kc_G\}, & \text{if } G \neq G_2; \\ \max\{3\ell(w_0) + 3k - 3, 2kc_G\}, & \text{if } G = G_2. \end{cases}$$

For the interested reader, a straightforward type-by-type calculation using the values of c_G from Definition 5.9 as well as Property (2) from Proposition 3.1 which says that $k \leq \ell(w_0)$ leads to the following explicit calculation:

$$(5.15) \quad M_k = \begin{cases} 2\ell(w_0) + 2k - 2 & \text{if } G = A_n, \\ 2\ell(w_0) + 2k - 2 & \text{if } G = B_n, C_n, \text{ or } D_n \text{ and } k \leq \ell(w_0) - 1, \\ 4k & \text{if } G = B_n, C_n, \text{ or } D_n \text{ and } k > \ell(w_0) - 1, \\ 2\ell(w_0) + 2k - 2 & \text{if } G = E_6 \text{ and } k \leq \frac{\ell(w_0)-1}{2}, \\ 6k & \text{if } G = E_6 \text{ and } k > \frac{\ell(w_0)-1}{2}, \\ 2\ell(w_0) + 2k - 2 & \text{if } G = E_7 \text{ or } F_4 \text{ and } k \leq \frac{\ell(w_0)-1}{3}, \\ 8k & \text{if } G = E_7 \text{ or } F_4 \text{ and } k > \frac{\ell(w_0)-1}{3}, \\ 2\ell(w_0) + 2k - 2 & \text{if } G = E_8 \text{ and } k \leq \frac{\ell(w_0)-1}{5}, \\ 12k & \text{if } G = E_8 \text{ and } k > \frac{\ell(w_0)-1}{5}, \\ 3\ell(w_0) + 3k - 3 & \text{if } G = G_2 \text{ and } k \leq \ell(w_0) - 1, \\ 6k & \text{if } G = G_2 \text{ and } k > \ell(w_0) - 1. \end{cases}$$

Recall that, in addition to a superregularity hypothesis, part (ii) of Proposition 4.5 concerning paths in the quantum Bruhat graph also placed a mild hypothesis on the length of x . Our next sublemma demonstrates that this length condition is actually implied by the new superregularity hypothesis on λ in terms of M_k .

Sublemma 5.11. *Let $x = t^{v\lambda}w \in \widetilde{W}$, and suppose that $k \geq 1$ is the length of any minimal path from $w^{-1}v$ to v in the quantum Bruhat graph for W . If $\langle \lambda, \alpha_j \rangle > M_k$ for all simple roots $\alpha_j \in \Delta$, then $\ell(x) > k$.*

Proof. First recall the formula for $\ell(x)$ from Lemma 4.1, which we bound from below as follows:

$$(5.16) \quad \ell(x) = \langle \lambda, 2\rho \rangle - \ell(w^{-1}v) + \ell(v) \geq \langle \lambda, 2\rho \rangle - \ell(w_0).$$

Since R is a reduced root system, then for any simple root $\alpha_j \in \Delta$, we know that $\langle \lambda, 2\rho, \rangle \geq \langle \lambda, \alpha_j \rangle$. We can thus see that

$$(5.17) \quad \ell(x) \geq \langle \lambda, 2\rho \rangle - \ell(w_0) \geq \langle \lambda, \alpha_j \rangle - \ell(w_0) > M_k - \ell(w_0)$$

by our superregularity hypothesis. Using the definition of M_k , we can further deduce that

$$(5.18) \quad \ell(x) > M_k - \ell(w_0) \geq (2\ell(w_0) + 2k - 2) - \ell(w_0) = \ell(w_0) + 2k - 2.$$

But since both $\ell(w_0) \geq 1$ and $k \geq 1$, then

$$(5.19) \quad \ell(x) > \ell(w_0) + 2k - 2 \geq k,$$

as required. \square

5.5. Root hyperplanes for maximal translations. We are now prepared to present our final lemma, which represents both a culmination of the collection of technical results in this section and the crux of the argument in the second of two key propositions. In words, Lemma 5.12 guarantees that the Newton point for any translation immediately below x in Bruhat order lies outside of the convex hull of the Newton point of x itself, an observation that is critical in the proof of Proposition 6.1 in the next section.

Lemma 5.12. *Let $x = t^\lambda w \in \widetilde{W}$, and assume that $x \succ x_1 \succ x_2 \succ \cdots \succ x_k = t^\mu$ is any minimal length saturated chain from x to a pure translation, where $k \geq 1$. Suppose that $\langle \lambda, \alpha_j \rangle > M_k$ for all simple roots $\alpha_j \in \Delta$. Then for any $1 \leq i \leq n$ such that $\nu_i(x) \geq \nu(x)$, we have*

$$(5.20) \quad F_i(\mu^+) > F_i(\nu_i(x)).$$

Proof. Since $k \geq 1$, then we know that the finite part $w \neq 1$. In addition, the superregularity hypothesis on λ together with the assumption that $k \geq 1$ imply that λ is dominant. Therefore, Lemma 5.3 says there exists $1 \leq i \leq n$ such that $\nu_i(x) \geq \nu(x)$.

Note by Sublemma 5.11 that the hypotheses of Proposition 4.5 apply under our current superregularity assumption. Since x and t^μ differ by a sequence of affine reflections, Proposition 4.5 says that $\mu^+ = \lambda - \sum \beta^\vee$, where this sum is a nonnegative combination of at most k positive coroots $\beta^\vee \in R^\vee$, corresponding to the downward edges in any minimal length path from w^{-1} to 1 in the quantum Bruhat graph.

We now define an element $\mu' = \lambda - k\tilde{\mu} \in Q^\vee$, where $\tilde{\mu}$ is the highest coroot in R^\vee . The element μ' is designed to be a lower bound for μ^+ in the following sense:

$$(5.21) \quad \mu^+ - \mu' = \left(\lambda - \sum \beta^\vee \right) - (\lambda - k\tilde{\mu}) = k\tilde{\mu} - \sum \beta^\vee.$$

Since there are at most k summands in the rightmost sum, and for each of them we know that $\tilde{\mu} \geq \beta^\vee$, then this difference is a nonnegative linear combination of coroots. Therefore, $\mu^+ - \mu' \geq 0$, and so by Lemma 5.8 and linearity we have $F_i(\mu^+ - \mu') \geq 0$ and thus $F_i(\mu^+) \geq F_i(\mu')$.

To prove the current lemma, it therefore suffices to verify that $F_i(\mu') > F_i(\nu_i(x))$. Compute that

$$(5.22) \quad F_i(\mu') = F_i(\lambda - k\tilde{\mu}) = F_i(\lambda) - kF_i(\tilde{\mu}).$$

Now recall that $\nu_i(x)$ is the projection of λ onto the hyperplane H_{α_i} , which means that $\nu_i(x) = \lambda - \frac{1}{2}\langle \lambda, \alpha_i \rangle \alpha_i^\vee$. Therefore,

$$(5.23) \quad F_i(\nu_i(x)) = F_i(\lambda) - \frac{\langle \lambda, \alpha_i \rangle}{2} F_i(\alpha_i^\vee).$$

Taking the difference of the two previous equations and using the positivity in Lemma 5.8, we see that $F_i(\mu') > F_i(\nu_i(x))$ if and only if

$$(5.24) \quad kF_i(\tilde{\mu}) < \frac{\langle \lambda, \alpha_i \rangle}{2} F_i(\alpha_i^\vee).$$

Our next aim is thus to compare the values $F_i(\tilde{\mu})$ and $F_i(\alpha_i^\vee)$.

Expanding $\tilde{\mu}$ in terms of the basis of simple coroots as in Table 1, we have $\tilde{\mu} = c_1\alpha_1^\vee + \cdots + c_n\alpha_n^\vee$ for some $c_i \in \mathbb{Z}_{\geq 0}$. Therefore,

$$(5.25) \quad F_i(\tilde{\mu}) = F_i(c_1\alpha_1^\vee + \cdots + c_n\alpha_n^\vee) = \sum_{j=1}^n c_j F_i(\alpha_j^\vee) = c_i F_i(\alpha_i^\vee),$$

since by construction $F_i(\alpha_j^\vee) = 0$ for all $i \neq j$. For each root system we have $c_i \leq c_G$ by the definition of c_G . Therefore, we have

$$(5.26) \quad F_i(\tilde{\mu}) \leq c_G F_i(\alpha_i^\vee).$$

By our superregularity hypothesis on λ , we know that $\langle \lambda, \alpha_i \rangle > 2kc_G$ for any G . Putting these observations together, we see that

$$(5.27) \quad kF_i(\tilde{\mu}) \leq kc_G F_i(\alpha_i^\vee) = \frac{2kc_G}{2} F_i(\alpha_i^\vee) < \frac{\langle \lambda, \alpha_i \rangle}{2} F_i(\alpha_i^\vee),$$

and so $F_i(\mu^+) \geq F_i(\mu') > F_i(\nu_i(x))$, as desired. \square

6. TRANSLATIONS DOMINATE NEWTON POINTS

We are now prepared to present our second of two key propositions, followed by the proof of our main theorem. Recall Theorem 2.10 which says that to compute ν_x we should take a maximum in dominance order over all $\nu(y)$ such that $x \geq y$. The simple hint suggested by Lemma 5.2 and further supported by Lemma 5.12, is that we can in fact reduce the problem to focusing exclusively on the pure translations. Proposition 6.1 makes this reduction step precise, and the proof of Theorem 3.2 follows immediately in Section 6.2. We conclude in Section 6.3 with a discussion of the superregularity hypotheses required at various steps in the proof.

6.1. Reduction to pure translations. Our second key proposition says that in order to compute the generic Newton point in IxI , it suffices to find any largest pure translation element which is less than or equal to x in Bruhat order. The proof uses critically both the correspondence to paths in the quantum Bruhat graph established in Proposition 4.5, in addition to the entire sequence of lemmas from Section 5.

Proposition 6.1. *Let $x = t^{v\lambda}w \in \widetilde{W}$, and consider any minimal length saturated chain in Bruhat order from x to a pure translation, say $x \succ x_1 \succ \cdots \succ x_k = t^\mu$. If $\langle \lambda, \alpha_j \rangle > M_k$ for all simple roots $\alpha_j \in \Delta$, then $\nu_x = \mu^+$.*

Proof. This proof proceeds by induction on the length k of the saturated chain. If $k = 0$, then $x = t^{v\lambda} = t^\mu$ is already a pure translation, and so Proposition 2.5 says that $\nu(x) = \lambda^+$. We also know by Mazur's inequality that $\nu_x \leq \lambda^+$. Theorem 2.10 says that $\nu_x = \max\{\nu(y) \mid y \leq x\}$, and so clearly $\lambda^+ \geq \nu_x \geq \nu(x) = \lambda^+$, concluding the base case.

Now consider any minimal length saturated chain from x to a pure translation of length $k \geq 1$, say $x \succ x_1 \succ \cdots \succ x_k = t^\mu$. Clearly, the truncation $x_1 \succ x_2 \succ \cdots \succ x_k = t^\mu$ is a saturated chain of length $k - 1$. Because x_1 is a cocover of x , Using Proposition 4.2, we can explicitly write $x_1 = t^{v'\lambda'} r_{v\alpha} w$ for some $\alpha \in R^+$. In any of the four cases in Proposition 4.2, either $\lambda' = \lambda$ or $\lambda' = \lambda - \alpha^\vee \in Q^+$. Using our superregularity hypothesis, we can compute that for any $\alpha_i \in \Delta$

$$(6.1) \quad \langle \lambda', \alpha_i \rangle \geq \langle \lambda - \alpha^\vee, \alpha_i \rangle > \begin{cases} M_k - 2 = \max\{2\ell(w_0) + 2k - 2, 2kc_G\} - 2, & \text{if } G \neq G_2, \\ M_k - 3 = \max\{3\ell(w_0) + 3k - 3, 2kc_G\} - 3, & \text{if } G = G_2. \end{cases}$$

Note that $(2\ell(w_0) + 2k - 2) - 2 = 2\ell(w_0) + 2(k - 1) - 2$, and similarly for $G = G_2$. On the other hand, since $c_G \geq 1$, we also have that $2kc_G - 2 \geq 2(k - 1)c_G$. These calculations demonstrate that $M_k - 2 \geq M_{k-1}$ if $G \neq G_2$, and similarly for $G = G_2$. The superregularity condition $\langle \lambda', \alpha_i \rangle > M_{k-1}$

thus permits us to apply the induction hypothesis to the saturated chain $x_1 \succ x_2 \succ \cdots \succ x_k = t^\mu$ of length $k - 1$, from which we conclude that $\nu_{x_1} = \mu^+$ and therefore $\nu_x \geq \mu^+$.

Now consider any $y \in \widetilde{W}$ such that $y < x$. We have two cases: either y occurs in some saturated chain of minimal length between x and a pure translation, or it occurs later in such a chain. In the first case we have $x \succ x'_1 \succ \cdots \succ y \succ \cdots \succ x'_k = t^\gamma$ for some $\gamma \in Q^\vee$. By the inductive hypothesis applied to the chain beginning at x'_1 , we know that $\nu_{x'_1} = \gamma^+$. Therefore, in this case we see that $\nu(y) \leq \gamma^+ \leq \nu_x$. In the second case, we can still place y in a saturated chain of the form $x \succ x'_1 \succ \cdots \succ x'_k = t^\gamma \succ \cdots \succ y$. Again use the fact that $\nu_{x'_1} = \gamma^+ = \max\{\nu(z) \mid z \leq x'_1\}$ to see that $\nu(y) \leq \gamma^+ \leq \nu_x$. In both cases, for any $y < x$, we have $\nu(y) \leq \gamma^+$.

Theorem 2.10 and uniqueness of ν_x now imply that either $\nu_x = \nu(x)$ or $\nu_x = \mu^+$ for some maximal translation t^μ such that $t^\mu \leq x$. Our next goal is thus to compare $\nu(x)$ and μ^+ for any t^μ as in the statement of the proposition. In particular, we aim to prove that $\nu(x) \not\geq \mu^+$, in which case $\nu_x \neq \nu(x)$. Since $k \geq 1$, we know that x itself is not a translation, and therefore $\nu(x) \neq \mu^+$. The element $x = t^{v\lambda}w$ might not lie in the dominant Weyl chamber, but recall from Lemma 5.5 that $\nu(x) = \nu(t^\lambda v^{-1}wv)$. For the purpose of computing the Newton point, we can thus replace x by $x' = t^\lambda v^{-1}wv$. Recall that $\lambda \in Q^+$, and note that $w \neq 1$ if and only if $v^{-1}wv \neq 1$. We may therefore apply Lemma 5.3 to x' to obtain an index $1 \leq i \leq n$ such that $\nu_i(x) = \nu(t^\lambda s_i) \geq \nu(x') = \nu(x)$. Of course, if it were the case that $\nu(x) \geq \mu^+$, then $\nu_i(x) \geq \nu(x) \geq \mu^+$ as well.

We proceed to relate the elements $\nu_i(x)$ and μ^+ . Recall Lemma 5.1, which reinterprets dominance order in terms of convexity. The point $\nu_i(x)$ sits on the hyperplane H_{α_i} , which is a wall of the dominant Weyl chamber. Since both $\nu_i(x)$ and μ^+ are points in the closed dominant Weyl chamber, then Lemma 5.1 says that $\nu_i(x) \geq \mu^+$ if and only if $\mu^+ \in \text{Conv}(W\nu_i(x))$. On the other hand, one face of $\text{Conv}(W\nu_i(x))$ which is contained in the dominant Weyl chamber is spanned precisely by the simple roots in $\Delta_P = \Delta \setminus \{\alpha_i\}$, each shifted by the vector $\nu_i(x)$. That is, this face of $\text{Conv}(W\nu_i(x))$ is contained in an affine translate of the hyperplane $H_{\widehat{\alpha}_i}$ defined in Lemma 5.7, shifted so that $\nu_i(x)$ is the origin. The hyperplane $H_{\widehat{\alpha}_i} + \nu_i(x)$ is therefore determined precisely by the linear functional $F_i : \mathbb{R}^r \rightarrow \mathbb{R}$ of Lemma 5.7, which means that $\vec{v} \in H_{\widehat{\alpha}_i} + \nu_i(x)$ if and only if $F_i(\vec{v}) = F_i(\nu_i(x))$. In particular, since $\text{Conv}(W\nu_i(x))$ contains the origin, if $F_i(\vec{v}) > F_i(\nu_i(x))$, then $\vec{v} \notin \text{Conv}(W\nu_i(x))$. Since by hypothesis $\langle \lambda, \alpha_j \rangle > M_k$ for all $\alpha_j \in \Delta$, we can apply Lemma 5.12, which says that $F_i(\mu^+) > F_i(\nu_i(x))$. Therefore $\mu^+ \notin \text{Conv}(W\nu_i(x))$. By Lemmas 5.1 and 5.7, we thus have that $\nu_i(x) \not\geq \mu^+$, and hence we can also conclude that $\nu(x) \not\geq \mu^+$. Since $\nu_x = \max\{\nu(y) \mid y \leq x\}$, we know that $\nu_x \neq \nu(x)$.

In summary, so far we have shown that the maximum Newton point $\nu_x = \mu^+$ for some translation t^μ such that $x \succ x_1 \succ \cdots \succ x_k = t^\mu$ is a saturated chain of minimal length. On the other hand, by Proposition 3.1, we know that all shortest paths between any pair of elements in the quantum Bruhat graph have the same weight. Note that the hypotheses of Proposition 4.5 are met by our superregularity condition and Sublemma 5.11. Applying Propositions 3.1 and 4.5, if there are saturated chains of minimal length from x to both t^μ and t^γ , then in fact $\mu^+ = \gamma^+$. It thus suffices to choose any of these translations in order to compute $\nu_x = \mu^+$. \square

6.2. Proof of the main theorem. We are finally prepared to complete the proof of our main result, and so we provide a brief reminder of the theorem statement. Given an affine Weyl group element $x = t^{v\lambda}w$, we start by considering any path of minimal length k from $w^{-1}v$ to v in the quantum Bruhat graph for W . Under the superregularity hypothesis $\langle \lambda, \alpha_j \rangle > M_k$ for all simple roots $\alpha_j \in \Delta$, we must prove that the maximum Newton point in $N(G)_x$ equals $\nu_x = \lambda - \alpha_x^\vee$, where α_x^\vee is the weight of the chosen path from $w^{-1}v$ to v . The proof now follows nearly directly from the two key Propositions 4.5 and 6.1.

Proof of Theorem 3.2. First recall by Sublemma 5.11 that the superregularity hypothesis in Theorem 3.2 implies the hypotheses required to apply Proposition 4.5. Use Proposition 4.5 (ii) to say

that the chosen path of minimal length k from $w^{-1}v$ to v can be lifted to 2^k different minimal length saturated chains in Bruhat order of the form $x \succ x_1 \succ \cdots \succ x_k = t^\mu$. Choosing any of these chains, we can now directly apply Proposition 6.1, which says that $\nu_x = \mu^+$.

To find the value of μ^+ , look at each covering relation in the sequence $x \succ x_1 \succ \cdots \succ x_k = t^\mu$, and note by Proposition 4.2 that the dominant element in the W -orbit of the translation part changes if and only if $x_j \succ x_{j+1}$ is a covering relation of type (2) or (4). By Remark 4.3, the edges of type (2) and (4) in this chain correspond precisely to the downward edges in the path from $w^{-1}v$ to v . The weight of a path in the quantum Bruhat graph is defined by summing the coroots coming from exactly the downward edges, and therefore $\mu^+ = \lambda - \alpha_x^\vee$, as desired. \square

6.3. Remarks on superregularity. Having proved our main result, we conclude with some remarks about each step in the proof which involves a superregularity hypothesis. For the reader interested in using Theorem 3.2 to compute smaller examples, we point out several ways in which one expects savings from the bound $\langle \lambda, \alpha_j \rangle > M_k$ in practice. For clarity of the exposition, we restrict ourselves to the case $G \neq G_2$, although all comments apply equally to $G = G_2$ with the constants appropriately adjusted.

The first main superregularity hypothesis is introduced in Proposition 4.2, which requires that $\langle \lambda, \alpha_j \rangle > 2\ell(w_0)$. The coefficient is forced by noting that the maximum value of $\langle \alpha^\vee, \alpha_j \rangle$ for any $\alpha \in R^+$ and $\alpha_j \in \Delta$ is attained already in the important special case $\alpha = \alpha_j$, but the factor of $\ell(w_0)$ is a softer bound. The basic principle of the proof uses $\ell(t^\lambda)$ as a proxy for $\ell(x)$, but this estimate is off by exactly $\ell(w^{-1}v) - \ell(v)$. Since this difference can actually equal $\ell(w_0)$ in very special cases, the bound is forced by the desire to have a uniform proof for all pairs $w, v \in W$. However, in practice this bound is obviously stronger than necessary.

The subsequent need to iterate Proposition 4.2 introduces yet another superregularity criterion. This additional linear term of $2k - 2$ comes about from the fact that the proofs of the two main Propositions 4.5 and 6.1 are inductive on k and thus apply Proposition 4.2 exactly k times. However, the assumption made in those proofs, which is once again stronger than necessary, is that *every* step in the chosen path from $w^{-1}v$ to v is a downward edge. In such circumstances, one does indeed subtract a coroot from λ at each step, but in practice these are again fairly special cases.

The final superregularity criterion $\langle \lambda, \alpha_j \rangle > 2kc_G$ is introduced in the course of Lemma 5.12. This additional hypothesis comes about as a result of using the worst-case-scenario upper bound for the weight of a path of minimal length k from $w^{-1}v$ to v in the quantum Bruhat graph, namely $k\tilde{\mu}$. In practice, this is an egregious bound, and the reader well-versed in the combinatorics of the quantum Bruhat graph can quite likely propose a sharp(er) upper bound on the weight of such a path. The analysis provided in Equation (5.15) shows, however, that the condition $\langle \lambda, \alpha_j \rangle > 2kc_G$ is *barely* stronger than the superregularity hypotheses already required by other components of the proof. For example, if G is classical, we have only truly strengthened the hypothesis in the case of the maximum value $k = \ell(w_0)$; however, in this case all edges are necessarily directly upward and there is no need to estimate the weight of the path. Therefore, in practice one can completely ignore this additional factor of $2kc_G$, unless one is truly interested in computing examples in the exceptional groups.

Finally, we point out that our request that $\langle \lambda, \alpha_j \rangle$ for *all* $\alpha_j \in \Delta$ is also overly cautious. In any given example, one need only be concerned with the roots α_j which actually *occur* as labels on the edges in the path from $w^{-1}v$ to v . In fact, Lemma 5.12 only requires the additional superregularity hypothesis $\langle \lambda, \alpha_i \rangle > 2kc_G$ for the *single* root direction associated to $\nu_i(x)$.

To provide a concrete example, if $G = SL_3$ and one traces each of these comments through, one concludes that $\langle \lambda, \alpha_j \rangle > 1$ for $j \in \{1, 2\}$ is the sharpest possible superregularity hypothesis required to execute the proof; compare the values of $M_k \in \{4, 6, 8\}$ for $k \in \{0, 1, 2\}$, respectively. And conversely, the statement of Theorem 3.2 does sometimes fail if $\langle \lambda, \alpha_j \rangle \in \{0, 1\}$ for at least one value of j . Although we are not yet prepared to formulate a precise conjecture, it is highly

probable that the weakest possible hypothesis one can place on the main theorem is roughly that x lie in the “shrunk” Weyl chambers, which is a condition that exactly excludes alcoves which sit between the hyperplanes $H_{\alpha_j,0}$ and $H_{\alpha_j,1}$ for any $\alpha_j \in \Delta$. See the discussion following the proof of Proposition 4.4 in [LS10] for additional evidence in this direction.

REFERENCES

- [AB83] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.
- [BDW96] Aaron Bertram, Georgios Daskalopoulos, and Richard Wentworth. Gromov invariants for holomorphic maps from Riemann surfaces to Grassmannians. *J. Amer. Math. Soc.*, 9(2):529–571, 1996.
- [Bea09] Elizabeth Beazley. Codimensions of Newton strata for $SL_3(F)$ in the Iwahori case. *Math. Z.*, 263(3):499–540, 2009.
- [Bea12] Elizabeth T. Beazley. Maximal Newton polygons via the quantum Bruhat graph. In *24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012)*, Discrete Math. Theor. Comput. Sci. Proc., AR, pages 899–910. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2012.
- [Ber97] Aaron Bertram. Quantum Schubert calculus. *Adv. Math.*, 128(2):289–305, 1997.
- [BFP99] Francesco Brenti, Sergey Fomin, and Alexander Postnikov. Mixed Bruhat operators and Yang-Baxter equations for Weyl groups. *Internat. Math. Res. Notices*, (8):419–441, 1999.
- [Bou02] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 4–6*. Elements of Mathematics (Berlin). Springer-Verlag, 2002. Translated from the 1968 French original by Andrew Pressley.
- [CdLOGP91] Philip Candelas, Xenia C. de la Ossa, Paul S. Green, and Linda Parkes. A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. *Nuclear Phys. B*, 359(1):21–74, 1991.
- [Cha00] Ching-Li Chai. Newton polygons as lattice points. *Amer. J. Math.*, 122(5):967–990, 2000.
- [Dem72] Michel Demazure. *Lectures on p -divisible groups*. Lecture Notes in Mathematics, Vol. 302. Springer-Verlag, Berlin-New York, 1972.
- [DL76] P. Deligne and G. Lusztig. Representations of reductive groups over finite fields. *Ann. of Math. (2)*, 103(1):103–161, 1976.
- [FM15] Evgeny Feigin and Makedonskyi. Generalized Weyl modules, alcove paths and Macdonald polynomials. arXiv:1512.03254, 2015.
- [FW04] W. Fulton and C. Woodward. On the quantum product of Schubert classes. *J. Algebraic Geom.*, 13(4):641–661, 2004.
- [GHR10] Ulrich Görtz, Thomas J. Haines, Robert E. Kottwitz, and Daniel C. Reuman. Affine Deligne-Lusztig varieties in affine flag varieties. *Compos. Math.*, 146(5):1339–1382, 2010.
- [GHN15] Ulrich Görtz, Xuhua He, and Sian Nie. \mathbf{P} -alcoves and nonemptiness of affine Deligne-Lusztig varieties. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(3):647–665, 2015.
- [Gör10] Ulrich Görtz. Affine Springer fibers and affine Deligne-Lusztig varieties. In *Affine flag manifolds and principal bundles*, Trends Math., pages 1–50. Birkhäuser/Springer Basel AG, Basel, 2010.
- [Gro74] Alexandre Grothendieck. *Groupes de Barsotti-Tate et cristaux de Dieudonné*. Les Presses de l’Université de Montréal, Montreal, Que., 1974. Séminaire de Mathématiques Supérieures, No. 45 (Été, 1970).
- [Hit10] Petra Hitzelberger. Kostant convexity for affine buildings. *Forum Math.*, 22(5):959–971, 2010.
- [Hor54] Alfred Horn. Doubly stochastic matrices and the diagonal of a rotation matrix. *Amer. J. Math.*, 76:620–630, 1954.
- [Hum90] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.
- [Kat79] Nicholas M. Katz. Slope filtration of F -crystals. In *Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. I*, volume 63 of *Astérisque*, pages 113–163. Soc. Math. France, Paris, 1979.
- [Kos73] Bertram Kostant. On convexity, the Weyl group and the Iwasawa decomposition. *Ann. Sci. École Norm. Sup. (4)*, 6:413–455 (1974), 1973.
- [Kot85] Robert E. Kottwitz. Isocrystals with additional structure. *Compositio Math.*, 56(2):201–220, 1985.
- [Kot97] Robert E. Kottwitz. Isocrystals with additional structure. II. *Compositio Math.*, 109(3):255–339, 1997.
- [LNS⁺15] Cristian Lenart, Satoshi Naito, Daisuke Sagaki, Anne Schilling, and Mark Shimozono. A uniform model for Kirillov-Reshetikhin crystals III: Nonsymmetric Macdonald polynomials at $t = 0$ and Demazure characters. arXiv:1511.00465, 2015.
- [LS10] Thomas Lam and Mark Shimozono. Quantum cohomology of G/P and homology of affine Grassmannian. *Acta Math.*, 204(1):49–90, 2010.

- [Lus78] George Lusztig. *Representations of finite Chevalley groups*, volume 39 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, R.I., 1978. Expository lectures from the CBMS Regional Conference held at Madison, Wis., August 8–12, 1977.
- [Man63] Ju. I. Manin. Theory of commutative formal groups over fields of finite characteristic. *Uspehi Mat. Nauk*, 18(6 (114)):3–90, 1963.
- [Maz72] B. Mazur. Frobenius and the Hodge filtration. *Bull. Amer. Math. Soc.*, 78:653–667, 1972.
- [MST15] Elizabeth Milićević, Petra Schwer, and Anne Thomas. Dimensions of affine Deligne-Lusztig varieties: a new approach via labeled folded alcove walks and root operators. arXiv:1504.07076, 2015.
- [OS13] Daniel Orr and Mark Shimozono. Specializations of nonsymmetric Macdonald-Koornwinder polynomials. arXiv:1310.0279, 2013.
- [Pet96] Dale Peterson. Lectures on quantum cohomology of G/P . M.I.T., 1996.
- [Pos01] Alexander Postnikov. Symmetries of Gromov-Witten invariants. In *Advances in algebraic geometry motivated by physics (Lowell, MA, 2000)*, volume 276 of *Contemp. Math.*, pages 251–258. Amer. Math. Soc., Providence, RI, 2001.
- [Pos05] Alexander Postnikov. Quantum Bruhat graph and Schubert polynomials. *Proc. Amer. Math. Soc.*, 133(3):699–709 (electronic), 2005.
- [Rap00] Michael Rapoport. A positivity property of the Satake isomorphism. *Manuscripta Math.*, 101(2):153–166, 2000.
- [Rap05] Michael Rapoport. A guide to the reduction modulo p of Shimura varieties. *Astérisque*, (298):271–318, 2005. Automorphic forms. I.
- [RR96] M. Rapoport and M. Richartz. On the classification and specialization of F -isocrystals with additional structure. *Compositio Math.*, 103(2):153–181, 1996.
- [ST97] Bernd Siebert and Gang Tian. On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator. *Asian J. Math.*, 1(4):679–695, 1997.
- [Ste98] John R. Stembridge. The partial order of dominant weights. *Adv. Math.*, 136(2):340–364, 1998.
- [Vie14] Eva Viehmann. Truncations of level 1 of elements in the loop group of a reductive group. *Ann. of Math.* (2), 179(3):1009–1040, 2014.
- [Wit95] Edward Witten. The Verlinde algebra and the cohomology of the Grassmannian. In *Geometry, topology, & physics*, Conf. Proc. Lecture Notes Geom. Topology, IV, pages 357–422. Int. Press, Cambridge, MA, 1995.
- [WN16] Hideya Watanabe and Satoshi Naito. A combinatorial formula expressing periodic R -polynomials. arXiv:1603.02778, 2016.

DEPARTMENT OF MATHEMATICS & STATISTICS, HAVERFORD COLLEGE, HAVERFORD, PA, 19041, USA

E-mail address: emilicevic@haverford.edu

Current address: Max-Planck-Institut für Mathematik, Vivatsgasse 7, Bonn 53111, Germany

E-mail address: emilicev@mpim-bonn.mpg.de